CHARACTERISING ERGODICITY

ALESSIO DEL VIGNA

In this short note we mainly follows [2] and some exercises of [1] to collect characterisations of ergodicity.

We start by establishing the notation and recalling some definitions. Throughout the note, $(M, \mathfrak{B}, \mu)$ is a probability measure space and $f : M \to M$ a measurable transformation.

Definition 1. We say that f is *ergodic* with respect to the measure μ if for all measurable sets $B \in \mathfrak{B}$ such that $f^{-1}B \subseteq B$ it holds that $\mu(B) = 0$ or $\mu(B) = 1$.

1. Ergodicity via invariant sets

Invariants sets play a fundamental role in ergodic theory. They are measurable sets B whose backward image under a transformation f is not necessarily contained in B itself, but does not differ "too much" from B. To state the formal definition, recall that the *symmetric difference* of two sets A and B is the set

$$A \triangle B := (A \setminus B) \cup (B \setminus A).$$

Definition 2. A measurable set $B \in \mathfrak{B}$ is *invariant* under f is $\mu(B \triangle f^{-1}B) = 0$.

Definition 3. A function $\psi : M \to \mathbb{R}$ is *invariant* is $\psi(x) = \psi(f(x))$ for μ-a.e. $x \in M$.

Note that a measurable set B is invariant if and only if its indicator function $\mathbb{1}_B$ is an invariant function.

Theorem 4. *Let $(M, \mathfrak{B}, \mu)$ be a probability measure space and $f : M \to M$ a measure-preserving transformation. The following conditions are equivalent:*

(i) *f is ergodic with respect to μ;*

(ii) *for every invariant set $B \in \mathfrak{B}$ we have $\mu(B) = 0$ or $\mu(B) = 1$;*

(iii) *for every measurable set $B \in \mathfrak{B}$ with $\mu(B) > 0$ we have $\mu\left(\bigcup_{n\geq 0} f^{-n}B\right) = 1$;*

(iv) *for every measurable sets $A, B \in \mathfrak{B}$ with $\mu(A) > 0$ and $\mu(B) > 0$ there exists a positive integer j such that $\mu(f^{-j}A \cap B) > 0$.*

Proof. ((i)⇒(ii)) Let $B \in \mathfrak{B}$ be an invariant set. We claim that $\mu(f^{-n}B \triangle B) = 0$ for every integer $n \geq 0$. We have the inclusion

$$f^{-n}B \triangle B = f^{-n}B \setminus B \cup B \setminus f^{-n}B \subseteq \bigcup_{j=0}^{n-1} f^{-(n-j)}B \setminus f^{-(n-j-1)}B \cup \bigcup_{j=0}^{n-1} f^{-j}B \setminus f^{-(j+1)}B =$$

$$= \bigcup_{j=0}^{n-1} \left(f^{-(j+1)}B \triangle f^{-j}B\right) = \bigcup_{j=0}^{n-1} f^{-j}\left(f^{-1}B \triangle B\right).$$

Thus, as f preserves the measure μ, we have $\mu(f^{-n}B\triangle B) \leq n \cdot \mu(f^{-1}B\triangle B) = 0$, and the claim is proved. Now let

$$B_\infty := \bigcap_{n=0}^{\infty} \bigcup_{j=n}^{\infty} f^{-j}B.$$

For every $n \geq 0$ we have

$$\mu\left(B\triangle \bigcup_{j=n}^{\infty} f^{-j}B\right) \leq \sum_{j=n}^{\infty} \mu(B\triangle f^{-j}B) = 0.$$

Since the sets $\bigcup_{j=n}^{\infty} f^{-j}B$ decrease as n increases we have $\mu(B\triangle B_\infty) = 0$, which implies $\mu(B) = \mu(B_\infty)$. Moreover

$$f^{-1}B_\infty = \bigcap_{n=0}^{\infty} \bigcup_{j=n}^{\infty} f^{-(j+1)}B = \bigcap_{n=0}^{\infty} \bigcup_{j=n+1}^{\infty} f^{-j}B = B_\infty,$$

which means that the set B_∞ is invariant. Thus from (i) it follows that $\mu(B_\infty)$, which is equal to $\mu(B)$, is either 0 or 1.

((ii)⇒(iii)) Let $B \in \mathfrak{B}$ with $\mu(B) > 0$ and set $B' := \bigcup_{j=1}^{\infty} f^{-j}B$. We have $f^{-1}B' \subseteq B'$ and since f preserves μ we also have $\mu(f^{-1}B') = \mu(B')$. Hence $\mu(f^{-1}B'\triangle B') = 0$ and (ii) implies that $\mu(B')$ is either 0 or 1. By construction $f^{-1}B \subseteq B'$ and $\mu(f^{-1}B) = \mu(B) > 0$, thus $\mu(B') = 1$.

((iii)⇒(iv)) Let $A \in \mathfrak{B}$ and $B \in \mathfrak{B}$ with positive measure. By (iii) we have $\mu\left(\bigcup_{j=1}^{\infty} f^{-j}A\right) = 1$, thus

$$0 < \mu(B) = \mu\left(B \cap \bigcup_{j=1}^{\infty} f^{-j}A\right) = \mu\left(\bigcup_{j=1}^{\infty} (B \cap f^{-j}A)\right).$$

Then there must exist a positive integer j such that $\mu(B \cap f^{-j}A) > 0$.

((iv)⇒(i)) Let $B \in \mathfrak{B}$ a set such that $f^{-1}B \subseteq B$ and suppose by contradiction that $0 < \mu(B) < 1$. Then for all integers $j \geq 0$ we would have

$$0 = \mu(B \cap (M \setminus B)) = \mu(f^{-j}B \cap (M \setminus B)),$$

which contradicts (iv) since $\mu(B) > 0$ and $\mu(M \setminus B) > 0$. □

2. Ergodicity via invariant functions

Theorem 5. *Let $(M, \mathfrak{B}, \mu)$ be a probability measure space and $f : M \to M$ a measure-preserving transformation. The following conditions are equivalent:*

(i) *f is ergodic with respect to μ;*

(ii) *every integrable invariant function $\psi : M \to \mathbb{R}$ is constant μ-a.e. in M;*

(iii) *for every integrable invariant function $\psi : M \to \mathbb{R}$ we have $\psi(x) = \int \psi \, d\mu$ for μ-a.e. $x \in M$;*

(iv) *every invariant function $\psi : M \to \mathbb{R}$ with $\psi \in L^2(M, \mathfrak{B}, \mu)$ is constant μ-a.e. in M.*

Proof. ((i)⇒(ii)) For k and $n \geq 0$ integers define

$$X(k, n) := \left\{x \in M \, : \, \frac{k}{2^n} \leq \psi(x) < \frac{k+1}{2^n}\right\} = \psi^{-1}\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right).$$

The function ψ is measurable, thus $X(k,n) \in \mathfrak{B}$ for every k and every n. Since

$$f^{-1}X(k,n) \triangle X(k,n) \subseteq \{x \in M \,:\, \psi(f(x)) \neq f(x)\}$$

and ψ is f-invariant it follows that $\mu(f^{-1}X(k,n) \triangle X(k,n)) = 0$. From Theorem 4-(ii), the ergodicitiy of f implies that $\mu(X(k,n)) = 0$ or $\mu(X(k,n)) = 1$ for every k and n. As ψ is integrable, then ψ is finite almost everywhere, which is equivalent to say that for each n

$$\psi^{-1}\mathbb{R} = \psi^{-1}\left(\bigcup_{k=-\infty}^{\infty}\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)\right) = \bigcup_{k=-\infty}^{\infty}\psi^{-1}\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right) = \bigcup_{k=-\infty}^{\infty} X(k,n)$$

is equal to M up to a zero-measure set. Thus $\sum_{k=-\infty}^{\infty}\mu(X(k,n)) = \mu(M) = 1$, which implies that there is a unique k_n for which $\mu(X(k_n,n)) = 1$. Let

$$Y \coloneqq \bigcap_{n=1}^{\infty} X(k_n, n),$$

so that $\mu(Y) = 1$. Since by construction ψ is constant on Y we have that ψ is constant μ-a.e.

((ii)$\Rightarrow$(iii)) The validity of this implication is obvious, also recalling that $\mu(M) = 1$.

((iii)$\Rightarrow$(iv)) This implication clearly holds, as if $\psi \in L^2(M, \mathfrak{B}, \mu)$ then ψ is also integrable.

((iv)$\Rightarrow$(i)) We actually show that (iv) implies the condition stated in Theorem 4-(ii), which is equivalent to (i). Consider a set $B \in \mathfrak{B}$ such that $\mu(f^{-1}B \triangle B) = 0$. The function $\mathbb{1}_B$ is invariant and it is clearly in $L^2(M, \mathfrak{B}, \mu)$. Hence (iv) implies that $\mathbb{1}_B$ is constant μ-a.e. on M. But then either $\mathbb{1}_B(x) = 0$ for μ-a.e. $x \in M$ or $\mathbb{1}_B(x) = 1$ for μ-a.e. $x \in M$, so that $\mu(B) = \int \mathbb{1}_B \, \mathrm{d}\mu$ is respectively equal to 0 or 1, as we wanted to prove. □

References

[1] M. Viana and K. Oliveira, *Foundations of Ergodic Theory*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 2016

[2] C. Walkden, *Ergodic Theory*, `https://personalpages.manchester.ac.uk/staff/charles.walkden/ergodic-theory/ergodic_theory.pdf`, 2018

Dipartimento di Matematica, Università di Pisa
Email address: `alessio.delvigna@dm.unipi.it`