THE FACTORIAL NUMBER SYSTEM

ALESSIO DEL VIGNA

In this note we give a brief introduction to the factorial number system, proving existence and uniqueness of the factorial representation of positive integers.

1. INTRODUCTION

Mixed-radix numeral systems are positional number systems in which the weights associated to each position do not form a geometric sequence and instead form a sequence in which each weight is an integral multiple of the previous one, but not by the same factor. In this note we consider the so-called *factorial number system* (the name has been introduced in [2]), in which the weights are the factorial of the positive integers [1, 4].

Definition 1. Let n be a positive integer. The *factorial base representation* of n is given by

$$n = a_1 \cdot 1! + a_2 \cdot 2! + \cdots + a_k \cdot k!$$

where $0 \leq a_j \leq j$ for each $j = 1, \ldots, k$ and $a_k \neq 0$.

Remark 2. Note that k is the largest integer satisfying $k! \leq n < (k+1)!$

In Section 2 we prove that each positive integer admits a unique factorial base representation. For instance, the representation of the number 2020 in the factorial number system is

$$2020 = 2 \cdot 6! + 4 \cdot 5! + 4 \cdot 4! + 0 \cdot 3! + 2 \cdot 2! + 0 \cdot 1!$$

The following procedure, presented in [3], is a fast and easy way of finding the digits of the factorial representation of a positive integer n. Start setting $q_1 = n$. If $j \geq 1$ we perform the Euclidean division between q_j and the radix $j+1$, yielding $q_j = q_{j+1}(j+1) + r_j$. Then r_j is the j-th digit of the factorial representation, that is $a_j = r_j$. The quotients form a strictly decreasing sequence of non-negative integers, so that at some point they become zero and the process terminates.

Remark 3. Note that we start dividing by 2, since dividing by 1 would always yields $a_0 = 0$, and we omit this digit as it has no effect on the representation of any positive integer.

Applying the above procedure to $n = 2020$ we have

$$\begin{aligned}
2020 &= 1010 \cdot 2 + 0 \\
1010 &= 336 \cdot 3 + 2 \\
336 &= 84 \cdot 4 + 0 \\
84 &= 16 \cdot 5 + 4 \\
16 &= 2 \cdot 6 + 4 \\
2 &= 0 \cdot 7 + 2
\end{aligned}$$

which yields the above representation.

2. Existence and uniqueness of the factorial representation

We start by proving a property of factorials. In the language of the factorial base representation of numbers, this property tells us what happens when we consider a number with factorial representation such that $a_j = j$ for each $j = 1, \ldots, k$ and we add 1. The same question for the base 10 representation of numbers would be the following: what happens if we take $\underbrace{99\cdots 9}_{k \text{ times}}$ and add 1?

Lemma 4. *For every positive integer k holds*

$$\sum_{j=1}^{k} j \cdot j! + 1 = (k+1)!$$

Proof. We argue by induction on $k \geq 1$. The case $k = 1$ is trivial. Now suppose that the identity holds for a given $k \geq 1$, then

$$\sum_{j=1}^{k+1} j \cdot j! + 1 = \sum_{j=1}^{k} j \cdot j! + 1 + (k+1) \cdot (k+1)! = (k+1)! + (k+1) \cdot (k+1)! = (k+2)!$$

The inductive step is thus completed. □

Remark 5. A key consequence of Lemma 4 is the following: fixed a positive integer k, a factorial representation with k summands represents integers not exceeding $(k+1)! - 1$.

Theorem 6. *Every positive integer n admits a unique factorial base representation.*

Proof. (Uniqueness) Consider $n = \sum_{j=1}^{k} a_j \cdot j!$ and $m = \sum_{j=1}^{h} b_j \cdot j!$, where $0 \leq a_j \leq j$ for every $j = 1, \ldots, k$ and $0 \leq b_j \leq j$ for every $j = 1, \ldots, h$. We first note that if $k \neq h$, say $k < h$, we have

$$n = \sum_{j=1}^{k} a_j \cdot j! < (k+1)! \leq h! \leq \sum_{j=1}^{h} b_j \cdot j! = m,$$

where the first inequality holds by Lemma 4. Hence we can assume $k = h$ and we have to prove that if $n = m$ then $a_j = b_j$ for every $j = 1, \ldots, k$. By contradiction, suppose that there exists an index r such that $a_r \neq b_r$ and suppose that r is the smallest index with this property, so that

$$\sum_{j=r}^{k} (a_j - b_j) \cdot j! = 0.$$

If $r = k$ the thesis holds. If $r < k$, extracting the term with index r the remaining sum is a multiple of $(r+1)!$, so that

$$(a_r - b_r) \cdot r! + C \cdot (r+1)! = 0,$$

where C is integer. By construction $1 \leq |a_r - b_r| \leq r$, thus $|a_r - b_r| \cdot r! < (r+1)!$, which implies $C = 0$ and in turn $a_r = b_r$, a contradiction.

(Existence) Let k be a positive integer and consider factorial representation with k summands. We already observed in Remark 5 that in this way we can represent numbers n with $1 \leq n \leq (k+1)! - 1$. Moreover, the factorial representations are pairwise distinct by the first part of this proof and they are exactly $(k+1)! - 1$ because we have $j+1$ possible choices for the coefficient a_j and a_k cannot be zero. Thus we can apply the pigeonhole principle and conclude that each n with $1 \leq n \leq (k+1)! - 1$ has a factorial representation. Since k is arbitrary, the theorem is proved. □

We now give an alternative proof of the existence of the factorial representation arguing by induction on $n \geq 1$. The base step $n = 1$ is trivial. For the inductive step we suppose $n = \sum_{j=1}^{k} a_j \cdot j!$ with $0 \leq a_j \leq j$ for each $j = 1, \ldots, k$ and $a_k \neq 0$, and prove that also $n + 1$ admits a factorial representation. If $a_j = j$ for each $j = 1, \ldots, k$ then Lemma 4 implies $n + 1 = 1 \cdot (k + 1)!$, which is a factorial representation. Otherwise let r be the minimum index such that $a_r < r$, so that we can split

$$n = \sum_{j=1}^{r-1} j \cdot j! + \sum_{j=r}^{k} a_j \cdot j!$$

Hence

$$n + 1 = \sum_{j=1}^{r-1} j \cdot j! + 1 + \sum_{j=r}^{k} a_j \cdot j! = r! + \sum_{j=r}^{k} a_j \cdot j! = (a_r + 1) \cdot r! + \sum_{j=r+1}^{k} a_j \cdot j!,$$

which is a factorial representation because $1 \leq a_r + 1 \leq r$.

References

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Dipartimento di Matematica, Università di Pisa
Email address: alessio.delvigna@dm.unipi.it