1. Limiti

Es. 3

$$\lim_{(x,y)\to(0,0)} \underbrace{\left(\sin(x^2+y^2),\ \log(x^2+y^2)\right)}_{f(x,y)}$$

$$f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$\mathrm{Dom}(f) = \mathbb{R}^2 \setminus \{(0,0)\} \quad \Leftarrow \quad x^2+y^2>0 \iff (x,y)\neq(0,0)$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \infty \leftarrow \text{il p.to infin. di } \mathbb{R}^2$$

$$\Updownarrow$$

$$\lim_{(x,y)\to(0,0)} d_{\mathbb{R}^2}\left(f(x,y),(0,0)\right) = +\infty \leftarrow \text{il p.to inf. di } \mathbb{R}$$

$$= \sqrt{\underbrace{\sin^2(x^2+y^2)}_{\to 0} + \underbrace{\log^2(x^2+y^2)}_{\to +\infty}} \longrightarrow +\infty$$

Es. 9

$$\lim_{(x,y)\to(0,0)} \frac{x^3y^3}{x^3+y^3}$$

$$\mathrm{Dom}(f) = \{ y \neq -x \}$$

$x^3+y^3=0 \iff y=-x$

$$f|_{\{x=0\}}(y) = 0 \quad \forall y \neq 0$$

$$f|_{\{y=0\}}(x) = 0 \quad \forall x \neq 0$$

$$f|_{\{y=\lambda x\}}(x) = \frac{\lambda^3 x^{\cancel{6}\,3}}{(1+\lambda^3)\cancel{x^3}} \to 0 \quad \forall \lambda \neq -1$$

con $\lambda \neq -1$

$$f|_{\{y=-x+x^\beta\}}(x) = \frac{x^3(-x+x^\beta)^3}{x^3+(-x+x^\beta)^3} =$$

$\beta > 1$, $x>0$

$(-x+x^\beta)^3 =$

$= -x^3 + 3x^{2+\beta} +$

$- 3x^{1+2\beta} + x^{3\beta}$

$$= \frac{-x^6+o(x^6)}{\cancel{x^3}-\cancel{x^3}+3x^{2+\beta}+o(x^{2+\beta})}$$

verifica che $1+2\beta$ e 3β sono $> 2+\beta$ per $\beta>1$

Scegliamo $\beta = 4$:

$$= \frac{-x^6+o(x^6)}{3x^6+o(x^6)} \to -\frac{1}{3} \implies \text{il lim. } \nexists$$

Es. 12

$$\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^4+y^2} \qquad Dom(f) = \mathbb{R}^2 \setminus \{(0,0)\}$$

$$f|_{\{x=0\}}(y) = 0 \quad \forall y \neq 0$$

$$f|_{\{y=0\}}(x) = 0 \quad \forall x \neq 0$$

$$f|_{\{y=\lambda x\}}(x) = \frac{\lambda x^{\cancel{4}\,2}}{\cancel{x^2}(x^2+\lambda^2)} \to 0 \quad \forall \lambda \in \mathbb{R}$$

$$f|_{\{y=x^\alpha\}}(x) \to 0 \qquad \text{(ESERCIZIO)}$$

con $\alpha > 0$, $x > 0$

$$a^2+b^2 \geq 2ab$$

$$0 \leq \left| \frac{x^3 y}{x^4+y^2} \right| = \frac{|x|^3 |y|}{x^4+y^2} \leq \frac{|x|^{\cancel{3}} \cancel{|y|}}{2\cancel{x^2}\cancel{|y|}} =$$

$$= \frac{1}{2}|x| \to 0 \implies \text{il lim. è } 0$$

per il teorema del confronto

Es.

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^5}{x^2+y^4}$$

dis. triang.

$$0 \leq \left| \frac{x^3+y^5}{x^2+y^4} \right| = \frac{|x^3+y^5|}{x^2+y^4} \leq \frac{|x|^3+|y|^5}{x^2+y^4} =$$

$$= \frac{|x|^3}{x^2+y^4} + \frac{|y|^5}{x^2+y^4} =$$

$$= \underbrace{\frac{x^2}{x^2+y^4}}_{\leq 1} |x| + \underbrace{\frac{y^4}{x^2+y^4}}_{\leq 1} |y| \leq |x|+|y| \to 0$$

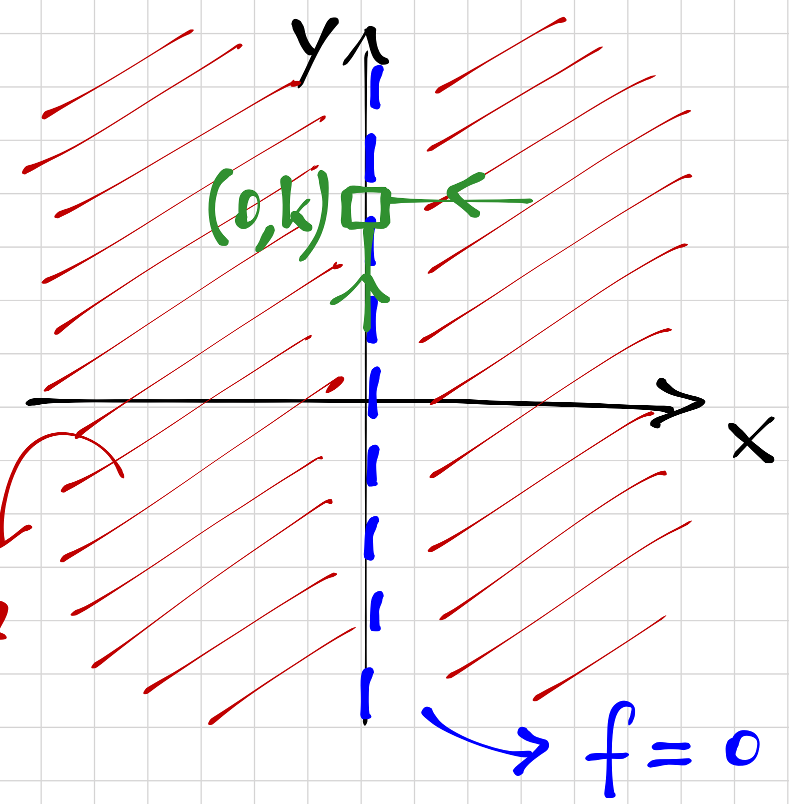
Es.

$$f(x,y) = \begin{cases} \dfrac{y^2}{x} & x \neq 0 \\ 0 & x = 0 \end{cases} \qquad Dom(f) = \mathbb{R}^2$$

(i) Continuità di f su $\mathbb{R}^2$.

(ii) Continuità di f su $D = \{ |y| \leq x \leq 1 \}$

(i) Ogni punto con $x \neq 0$ ha un intorno su cui f si scrive come quoziente di funz. continue $\Rightarrow f$ continua su $\mathbb{R}^2 \setminus \{x = 0\}$



Consideriamo i punti $(0,k)$:

$\lim_{(x,y)\to(0,k)} f(x,y)$ non esiste (vedi verde) per $k \neq 0$

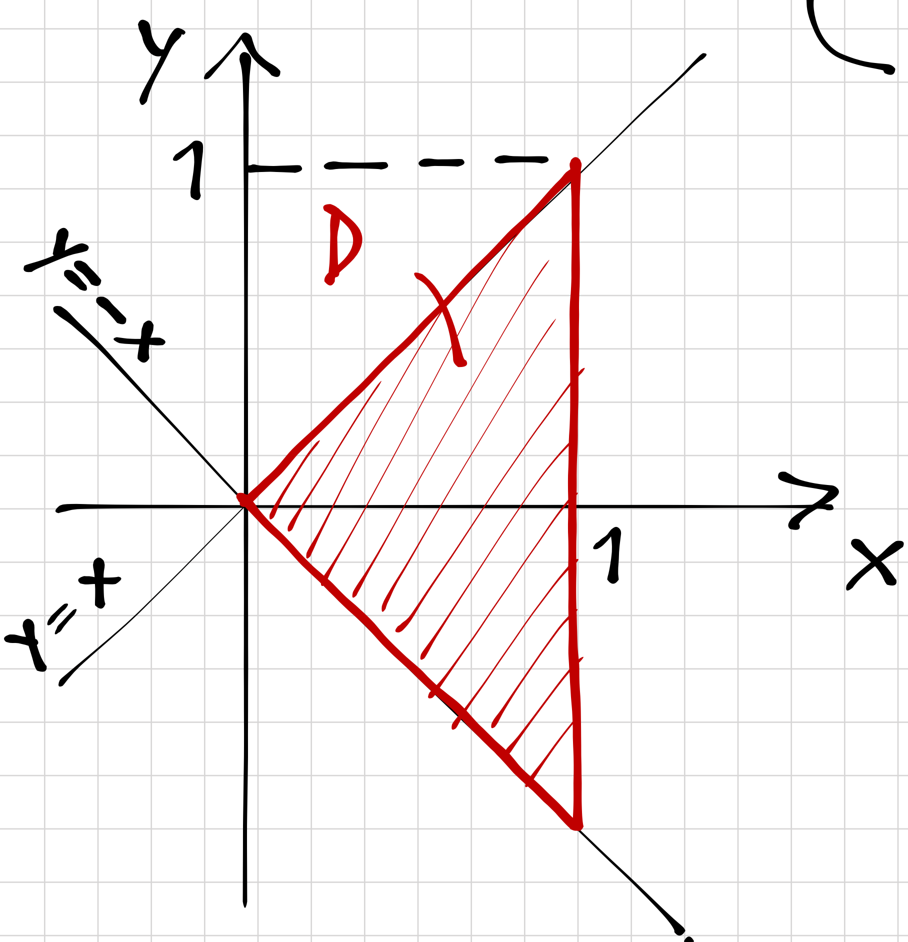
$\lim_{(x,y)\to(0,0)} f(x,y)$ $\qquad f|_{\{y=\sqrt{x}\}}(x) = 1 \quad \forall x \neq 0$

$\qquad f|_{\{y=0\}}(x) = 0 \quad \forall x$

$\Rightarrow$ il lim. $\nexists$ $\Rightarrow f$ continua su $\mathbb{R}^2 \setminus \{x = 0\}$

(ii) $D = \{ |y| \leq x \leq 1 \}$

$|y| \leq x \Leftrightarrow y \leq x \wedge y \geq -x$



(i)

$\Downarrow$

f continua in ogni p.to interno di D

(i) $\Rightarrow$ f continua in ogni p.to di $\partial D \setminus \{(0,0)\}$

Vediamo la continuità di $f|_D$ in $(0,0)$

$$\left. \begin{array}{l} f(0,0) = 0 \\ \displaystyle\lim_{\substack{(x,y)\to(0,0) \\ (x,y)\in D}} \frac{y^2}{x} = 0 \end{array} \right\} \Longrightarrow f|_D \text{ continua in } (0,0)$$

$$0 \leq \left| \frac{y^2}{x} \right| \underset{x \geq 0 \text{ in } D}{=} \frac{y^2}{x} \overset{|y| \leq x \text{ in } D}{\leq} \frac{x^2}{x} = x \longrightarrow 0$$

2. Calcolo differenziale

Es. 5

$$f(x,y) = \cos\left(\frac{\pi}{2} - xy\right)$$

$$v = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \qquad (x_0, y_0) = (0,1)$$

$$\lim_{t \to 0} \frac{f\big((x_0,y_0)+tv\big) - f(x_0,y_0)}{t} =$$

$$= \lim_{t \to 0} \frac{\cos\left(\frac{\pi}{2} - \frac{1}{2}t \cdot \left(1 + t\frac{\sqrt{3}}{2}\right)\right) \cancel{- \cos\frac{\pi}{2}}}{t} =$$

$$= \lim_{t \to 0} \frac{\sin\left(t/2 + t^2\sqrt{3}/4\right)}{t} =$$

$$= \lim_{t \to 0} \underbrace{\frac{\sin\left(t/2 + t^2\sqrt{3}/4\right)}{t/2 + t^2\sqrt{3}/4}}_{\to 1} \underbrace{\frac{t/2 + t^2\sqrt{3}/4}{t}}_{\to \frac{1}{2}} = \frac{1}{2}$$

$$\Longrightarrow \quad D_v f(x_0, y_0) = \frac{1}{2}$$

Es. 8

$$f(x,y) = (x+1)^2 - (y-1)^2 \sin x$$

$$(x_0, y_0) = (0,0) \quad , \quad v = (v_1, v_2)$$

f differenziabile su $\mathbb{R}^2$ perché nell'intorno di ogni punto di $\mathbb{R}^2$ f si scrive come somma/prod. di f. diff.

$$\underset{\text{Teo.}}{\Longrightarrow} \quad D_v f(x_0,y_0) = \langle \nabla f(x_0,y_0), v \rangle$$

$$\frac{\partial f}{\partial x}(x,y) = 2(x+1) - (y-1)^2 \cos x$$

$$\frac{\partial f}{\partial y}(x,y) = -2(y-1)\sin x$$

$$\nabla f(0,0) = \left(\frac{\partial f}{\partial x}(0,0), \frac{\partial f}{\partial y}(0,0) \right) = (1,0)$$

$$D_v f(0,0) = \langle \nabla f(0,0), v \rangle =$$

$$= \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\rangle = v_1$$

ESERCIZIO: farlo con la definizione

Es. 10

$$f(x,y,z) = \log_z (x^2 - y^2)$$

$$(x_0, y_0, z_0) = (\sqrt{e}, 0, e), \qquad v = (0,0,1)$$

$$\lim_{t \to 0} \frac{\log_{e+t} \quad - \log_e e}{t} =$$

$$= \lim_{t \to 0} \frac{1}{t} \left(\frac{\log_e e}{\log_e (e+t)} - 1 \right) =$$

$\log(e+t) = \log e\left(1+\frac{t}{e}\right) =$

$= 1 + \log\left(1+\frac{t}{e}\right)$

$$= \lim_{t \to 0} \frac{1}{t} \, \frac{1 - \log(e+t)}{\log(e+t)} =$$

$$= \lim_{t \to 0} \underbrace{\frac{-\log\left(1+\frac{t}{e}\right)}{\frac{t}{e}}}_{\to 1} \, \frac{1}{e \log(e+t)} = -\frac{1}{e} .$$