Calcolo differenziale

Es. 3(vi)

$$f(x,y)=\begin{cases}\dfrac{x^2y^3}{x^4+y^4} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$$

$\mathrm{Dom}(f)=\mathbb{R}^2$

f è diff. in $\mathbb{R}^2\setminus\{(0,0)\}$ per soliti teoremi

$$\frac{\partial f}{\partial x}(0,0)=\frac{\partial f}{\partial y}(0,0)=0$$

$$\lim_{(x,y)\to(0,0)}\underbrace{\frac{1}{\sqrt{x^2+y^2}}\,\frac{x^2y^3}{x^4+y^4}}_{g(x,y)}\ \nexists \Rightarrow f \text{ non diff. in } (0,0)$$

$$\Uparrow$$

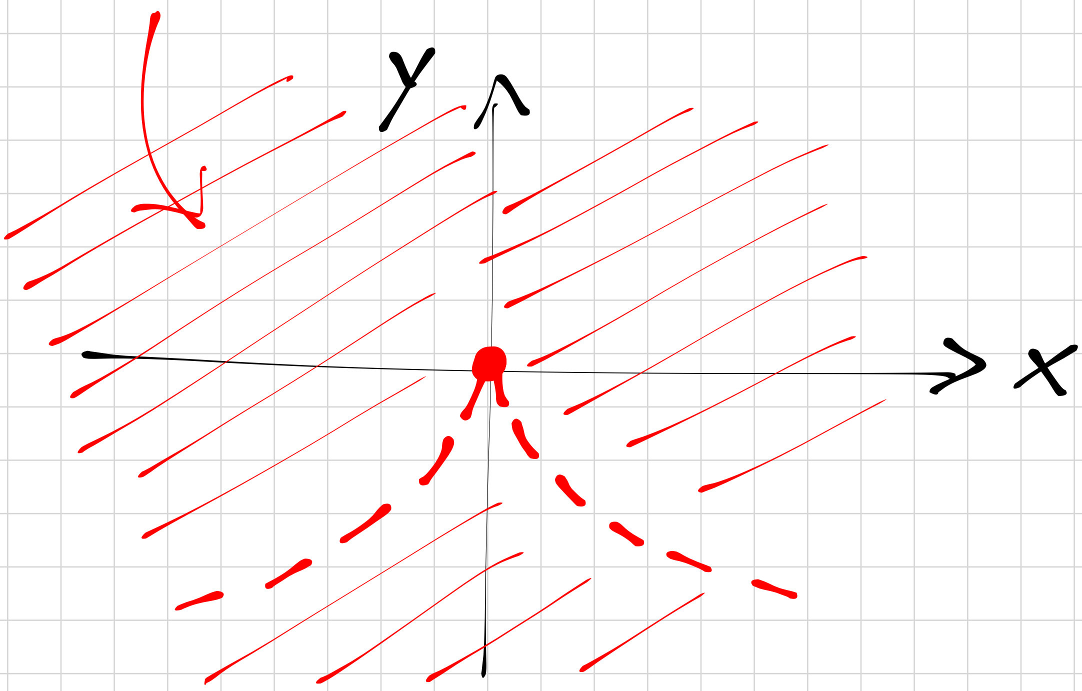
$$g\big|_{\{y=\lambda x\}}(x)=\frac{\lambda^3 \cancel{x^5}\,x}{|x|\sqrt{1+\lambda^2}\cdot\cancel{x^4}(1+\lambda^4)}$$

non ha lim. per $x\to 0$

Es. 3 (ix)

$$f(x,y)=\begin{cases} \dfrac{x^3+2xy^4}{x^2+y^3} & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$$

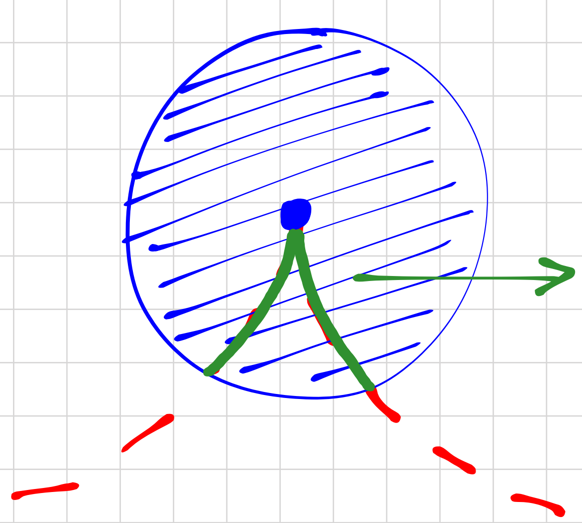
$$\mathrm{Dom}(f)=\left(\mathbb{R}^2 \setminus \{x^2+y^3=0\}\right)\cup\{(0,0)\}$$



$$x^2+y^3=0$$
$$\Updownarrow$$
$$y=-\sqrt[3]{x^2}$$

Nei p.ti di $\mathbb{R}^2\setminus\{x^2+y^3=0\}$ la funz. è diff. per soliti teoremi

$(0,0)$ non è p.to interno del $\mathrm{Dom}(f)$ $\Longrightarrow$ f non è diff. in $(0,0)$



$\notin \mathrm{Dom}(f)$

Es. 3

$$f(x,y) = (x-y)(x^2+y^2-1)$$

$$\mathrm{Dom}(f) = \mathbb{R}^2$$

$$\frac{\partial f}{\partial x}(x,y) = x^2+y^2-1+(x-y)\cdot 2x =$$
$$= 3x^2-2xy+y^2-1$$

$$\frac{\partial f}{\partial y}(x,y) = -x^2-y^2+1+(x-y)\cdot 2y =$$
$$= -x^2+2xy-3y^2+1$$

$$\nabla f(x,y) = (0,0) \iff \begin{cases} 3x^2-2xy+y^2-1=0 \\ -x^2+2xy-3y^2+1=0 \end{cases}$$

SOMMA

$$2x^2-2y^2=0$$

$$\iff x^2=y^2 \iff y=\pm x$$

$$\iff \begin{cases} y=x \\ 2x^2-1=0 \end{cases} \vee \begin{cases} y=-x \\ 6x^2-1=0 \end{cases} \iff$$

$$\iff \begin{cases} x=\pm 1/\sqrt{2} \\ y=\pm 1/\sqrt{2} \end{cases} \vee \begin{cases} x=\pm 1/\sqrt{6} \\ y=\mp 1/\sqrt{6} \end{cases}$$

$$H_f(x,y) = \begin{pmatrix} 6x-2y & -2x+2y \\ -2x+2y & 2x-6y \end{pmatrix}$$

$$P_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \qquad P_2 = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$$

$$H_f(P_1) = \begin{pmatrix} 4/\sqrt{2} & 0 \\ 0 & -4/\sqrt{2} \end{pmatrix} \Rightarrow P_1 \text{ sella}$$

$$H_f(P_2) = \begin{pmatrix} -4/\sqrt{2} & 0 \\ 0 & 4/\sqrt{2} \end{pmatrix} \Rightarrow P_2 \text{ sella}$$

$$P_3 = \left(\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right) \qquad P_4 = \left(-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

$$H_f(P_3) = \begin{pmatrix} 8/\sqrt{6} & -4/\sqrt{6} \\ -4/\sqrt{6} & 8/\sqrt{6} \end{pmatrix} = \frac{4}{\sqrt{6}} \underbrace{\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}}_{M}$$

$$\operatorname{Tr}(M) = 4, \quad \det(M) = 3$$

$$\Rightarrow 2 \text{ autoval.} > 0 \Rightarrow P_3 \text{ minimo loc.}$$

$$H_f(P_4) = \frac{4}{\sqrt{6}} \underbrace{\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}}_{M}$$

P_4 massimo loc.

$\Uparrow$

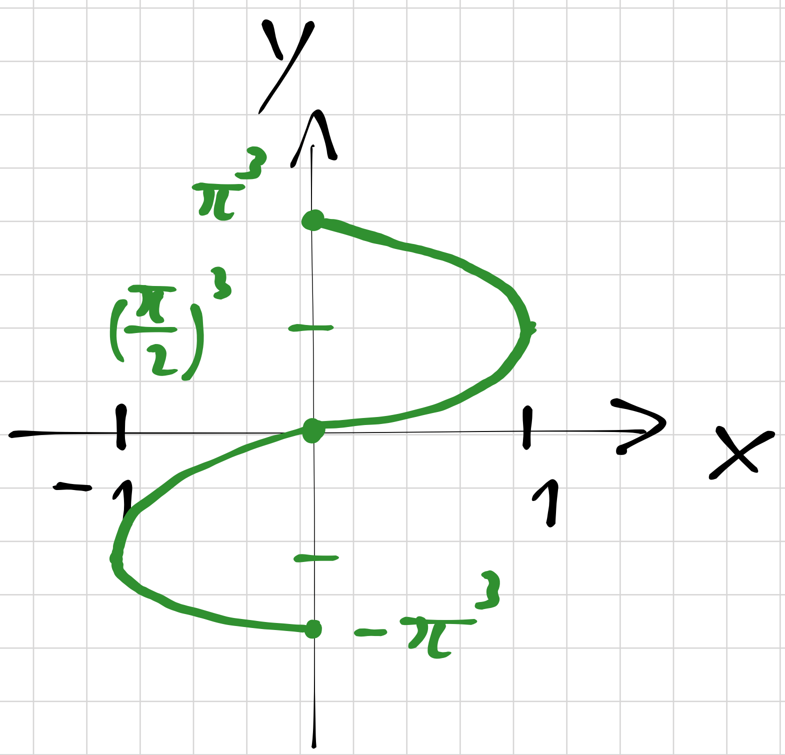
$$\operatorname{Tr}(M) = -4, \quad \det(M) = 3 \Rightarrow 2 \text{ autoval.} < 0$$

Es. 1

$\gamma : [-\pi, \pi] \to \mathbb{R}^2$

$\gamma(t) = (\sin t, t^3)$

$\gamma(-\pi) = (0, -\pi^3)$ $\quad \gamma(\pi) = (0, \pi^3)$



Int. con gli assi:

$\sin t = 0 \Leftrightarrow t = 0 \lor t = \pm\pi$

$\gamma(0) = (0,0)$

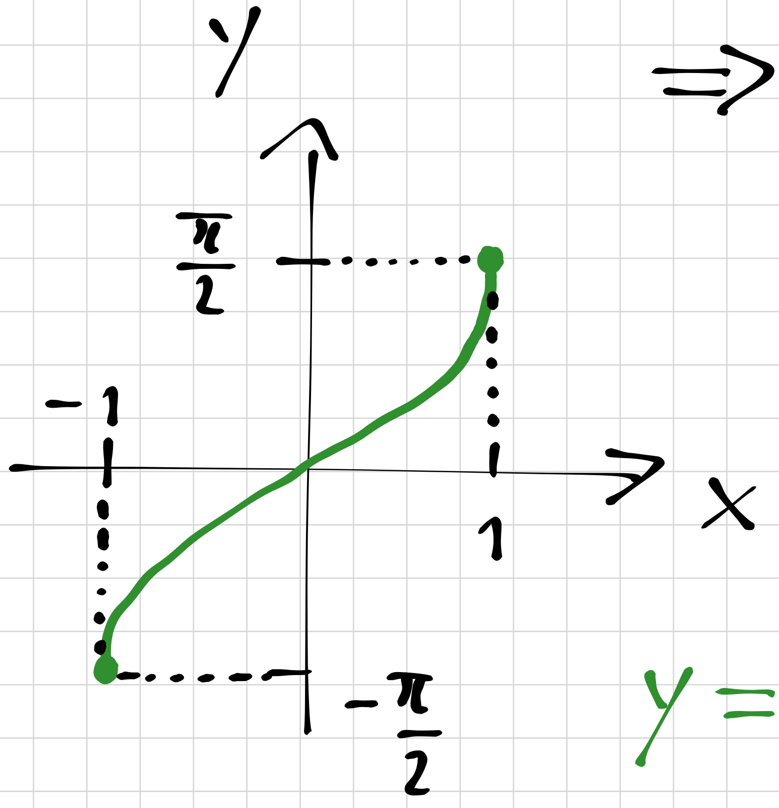
$$\begin{cases} x = \sin t \\ y = t^3 \end{cases}$$

Per $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ possiamo scrivere

$t = \arcsin x$

$\Rightarrow y = (\arcsin x)^3$

per $x \in [-1, 1]$



$y = \arcsin x$

Retta tang. in $P = (0,0)$

$\gamma'(t) = (\cos t, 3t^2)$

$P = (0,0) = \gamma(0)$

$\gamma'(0) = (1, 0)$

$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = P + \lambda \gamma'(0) = \begin{pmatrix} \lambda \\ 0 \end{pmatrix}$

$\Rightarrow \begin{cases} x = \lambda \\ y = 0 \end{cases}$ eq. parametriche della retta tang. al sost. di γ in P

$\Rightarrow \; y = 0$ eq. cartesiane

Es. 2

$\gamma : [0, 2\pi] \to \mathbb{R}^2$

$\gamma(t) = (t\cos t, t\sin t)$

$P = \left(0, \frac{\pi}{2}\right) = \gamma(t_\star)$

$$\begin{cases} t\cos t = 0 \\ t\sin t = \frac{\pi}{2} \end{cases} \overset{t\neq 0}{\iff} \begin{cases} \cos t = 0 \\ t\sin t = \frac{\pi}{2} \end{cases}$$

$$\iff \begin{cases} t = \pi/2 \\ \frac{\pi}{2}\cdot 1 = \pi/2 \end{cases} \vee \cancel{\begin{cases} t = \frac{3}{2}\pi \\ \frac{3}{2}\pi\cdot(-1) = \pi/2 \end{cases}}$$ non sol.

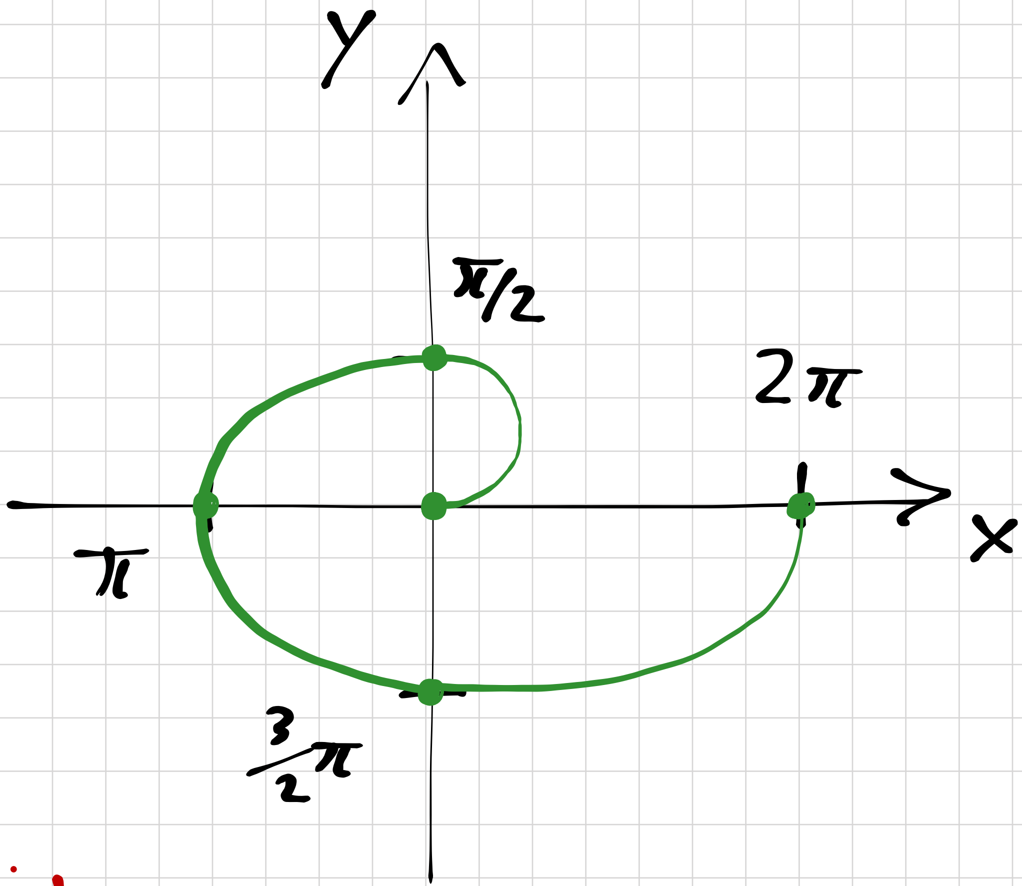
$$\Rightarrow P = \left(0, \frac{\pi}{2}\right) = \gamma\left(\frac{\pi}{2}\right)$$ $\to t_\star$

$\gamma'(t) = (\cos t - t\sin t, \sin t + t\cos t)$

$\gamma'\left(\frac{\pi}{2}\right) = \left(-\pi/2, 1\right)$

eq. param. $\begin{cases} x = -\lambda\,\pi/2 \\ y = \pi/2 + \lambda \end{cases}$

$\Rightarrow y = \frac{\pi}{2} - \frac{2}{\pi}x$ eq. cart.



$\gamma(0) = (0,0)$

$\gamma(2\pi) = (2\pi, 0)$

(i)

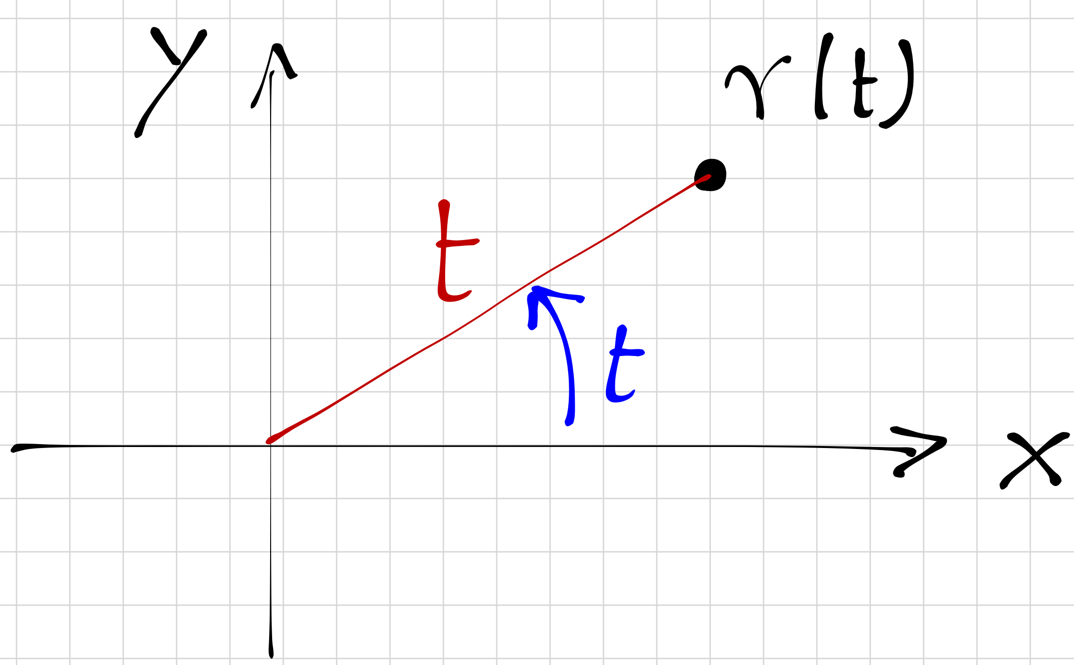
$$|\gamma(t)| = |(t\cos t, t\sin t)| = t$$

$\Rightarrow$ la dist. del p.to $\gamma(t)$ dall'origine aumenta con t

(ii)

$$\frac{y(t)}{x(t)} = \frac{\sin t}{\cos t} = \operatorname{tg} t$$

$\Rightarrow$ t è l'angolo che il p.to $\gamma(t)$ forma risp. a $(0,0)$



Es. 3

$\gamma : [0, 3\pi] \to \mathbb{R}^2$

$$\gamma(t) = \left(\frac{\sin t \cdot \cos t}{t}, \frac{\sin^2 t}{t} \right)$$

$$P = \left(0, \frac{2}{5\pi}\right) = \gamma(t_*)$$

$$\gamma(0) = \lim_{t \to 0^+} \gamma(t) = (1, 0)$$

$$\begin{cases} \dfrac{\sin t \cos t}{\cancel{t}} = 0 \\ \dfrac{\sin^2 t}{t} = \dfrac{2}{5\pi} \end{cases}$$

$t = \cancel{0}, \frac{\pi}{2}, \cancel{\pi}, \frac{3}{2}\pi, \cancel{2\pi}, \frac{5}{2}\pi, \cancel{3\pi}$ non sol. della II

$\Rightarrow t_* = \frac{5}{2}\pi$

$$\Rightarrow P = \gamma\left(\frac{5}{2}\pi\right)$$

$$\gamma'\left(\frac{5}{2}\pi\right) = \left(-\frac{2}{5\pi}, -\frac{4}{25\pi^2}\right)$$

$$\Rightarrow \begin{cases} x = -\lambda \frac{2}{5\pi} \\ y = \frac{2}{5\pi} - \lambda \frac{4}{25\pi^2} \end{cases} \Rightarrow y = \frac{2}{5\pi} + \frac{2}{5\pi} x$$

eq. cartesiana