Es.

$F: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ lineare definita da

$F\begin{pmatrix}2\\5\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$, $\quad F\begin{pmatrix}1\\3\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$

(i) $\mathcal{M}^{\mathcal{E}}_{\mathcal{E}}(F)$

(ii) $\mathcal{M}^{\mathcal{B}}_{\mathcal{B}'}(F)$ $\qquad \mathcal{B} = \left\{ \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\}$

$\mathcal{B}' = \left\{ \begin{pmatrix}2\\5\end{pmatrix}, \begin{pmatrix}1\\3\end{pmatrix} \right\}$

(i) $F\begin{pmatrix}1\\0\end{pmatrix} = F\left(3\begin{pmatrix}2\\5\end{pmatrix} - 5\begin{pmatrix}1\\3\end{pmatrix}\right) = (*)$

$\begin{pmatrix}1\\0\end{pmatrix} = a\begin{pmatrix}2\\5\end{pmatrix} + b\begin{pmatrix}1\\3\end{pmatrix}$

$$\begin{cases} 1 = 2a + b \\ 0 = 5a + 3b \end{cases} \Longleftrightarrow \begin{cases} a = 3 \\ b = -5 \end{cases}$$

$(*) = 3F\begin{pmatrix}2\\5\end{pmatrix} - 5F\begin{pmatrix}1\\3\end{pmatrix} = \begin{pmatrix}3\\-5\end{pmatrix}$

$F\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-1\\2\end{pmatrix}$ in modo analogo.

$$\Rightarrow \mathcal{M}^{\mathcal{E}}_{\mathcal{E}}(F) = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}$$

MODO ALTERNATIVO

$$F\begin{pmatrix}2\\5\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix} \qquad \mathrm{Imm}(f)=\mathbb{R}^2$$

$$F\begin{pmatrix}1\\3\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix} \qquad \Leftrightarrow \ker(f)=\{0\}$$

$$\underbrace{\Leftrightarrow F \text{ iniettiva}}$$

$$\Rightarrow F \text{ invertibile}$$

$$F^{-}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}2\\5\end{pmatrix}$$

$$F^{-1}\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}1\\3\end{pmatrix} \Rightarrow M^{\mathcal{E}}_{\mathcal{E}}(F^{-1})=\begin{pmatrix}2 & 1\\5 & 3\end{pmatrix}$$

$$M^{\mathcal{E}}_{\mathcal{E}}(F)=\left(M^{\mathcal{E}}_{\mathcal{E}}(F^{-1})\right)^{-1}=\begin{pmatrix}3 & -1\\-5 & 2\end{pmatrix}$$

(ii)

$$\begin{tikzcd}
\mathbb{R}^2_{\mathcal{B}} \arrow[r, "F"] \arrow[d, "id"'] & \mathbb{R}^2_{\mathcal{B}'} \\
\mathbb{R}^2_{\mathcal{E}} \arrow[r, "F"] & \mathbb{R}^2_{\mathcal{E}} \arrow[u, "id"']
\end{tikzcd}$$

$$\Rightarrow M^{\mathcal{B}}_{\mathcal{B}'}(F)=M^{\mathcal{E}}_{\mathcal{B}}(id)\,M^{\mathcal{E}}_{\mathcal{E}}(F)\,M^{\mathcal{B}'}_{\mathcal{E}}(id)$$

$$\mathcal{M}^{\mathcal{B}}_{\mathcal{C}}(id) = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix} \qquad \mathcal{B} = \left\{ \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\}$$

$$\mathcal{M}^{\mathcal{E}}_{\mathcal{B}'}(id) = \left(\mathcal{M}^{\mathcal{B}'}_{\mathcal{E}}(id) \right)^{-1} = \left(\begin{pmatrix}2 & 1\\ 5 & 3\end{pmatrix} \right)^{-1} =$$

$$= \begin{pmatrix}3 & -1\\ -5 & 2\end{pmatrix}$$

$$\begin{tikzcd} \mathbb{R}^2_{\mathcal{E}} \arrow[r, bend left, "id"] & \mathbb{R}^2_{\mathcal{B}'} \arrow[l, bend left, "id"] \end{tikzcd}$$

$$\mathcal{M}^{\mathcal{B}}_{\mathcal{B}'}(F) = \begin{pmatrix}3 & -1\\ -5 & 2\end{pmatrix}\begin{pmatrix}3 & -1\\ -5 & 2\end{pmatrix}\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix} =$$

$$= \begin{pmatrix}9 & -5\\ -16 & 9\end{pmatrix}$$

ESERCIZIO

Provare a calcolare $\mathcal{M}^{\mathcal{B}}_{\mathcal{B}'}(F)$ con la sua definizione.

Es. 2

$$V := \mathrm{Span}\left(\begin{pmatrix}1\\1\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\-3\\-3\\3\end{pmatrix}\right)$$

$$W_t := \mathrm{Span}\left(\begin{pmatrix}0\\t+1\\2t+1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\2\\1\end{pmatrix}\right), \ t \in \mathbb{R}$$

(i) $\dim V = 2$ perché generato da 2 vett. indip.

$\dim W_t = 2 \quad \forall t \in \mathbb{R}$, stesso motivo

(ii) $$V + W_t = \mathrm{Span}\left(\begin{pmatrix}1\\1\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\-3\\-3\\3\end{pmatrix}, \begin{pmatrix}0\\t+1\\2t+1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\2\\1\end{pmatrix}\right)$$

$$\begin{pmatrix} 1 & -1 & 0 & -1 \\ 1 & -3 & t+1 & 1 \\ 1 & -3 & 2t+1 & 2 \\ 0 & 3 & 1 & 1 \end{pmatrix} \xrightarrow[\substack{R_3 - R_1 \\ 2 \cdot R_4}]{R_2 - R_1}$$

$$\begin{pmatrix} 1 & -1 & 0 & -1 \\ 0 & -2 & t+1 & 2 \\ 0 & -2 & 2t+1 & 3 \\ 0 & 6 & 2 & 2 \end{pmatrix} \xrightarrow[R_4 + 3R_2]{R_3 - R_2}$$

$$\begin{pmatrix} 1 & -1 & 0 & -1 \\ 0 & -2 & t+1 & 2 \\ 0 & 0 & t & 1 \\ 0 & 0 & 3t+5 & 8 \end{pmatrix}$$

per convenienza scambiamo la III e IV colonna

$$\begin{pmatrix} 1 & -1 & -1 & 0 \\ 0 & -2 & 2 & t+1 \\ 0 & 0 & 1 & t \\ 0 & 0 & 8 & 3t+5 \end{pmatrix} \xrightarrow[R_4 - 8R_3]{}$$

$$\begin{pmatrix} 1 & -1 & -1 & 0 \\ 0 & -2 & 2 & t+1 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 5-5t \end{pmatrix}$$

$$t = 1 \Rightarrow \dim(V + W_t) = 3$$

$$\underset{(*)}{\Rightarrow} \dim(V \cap W_t) = 2 + 2 - 3 = 1$$

$$t \neq 1 \Rightarrow \dim(V + W_t) = 4$$

$$\underset{(*)}{\Rightarrow} \dim(V \cap W_t) = 2 + 2 - 4 = 0$$

(*) $\dim(V+W) = \dim(V) + \dim(W) - \dim(V \cap W)$

form. di Grassmann

(iii) $W_{-1} = \mathrm{Span}\left(\begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 2 \\ 1 \end{pmatrix} \right) \subseteq \mathbb{R}^4$

PRIMO MODO

$$\begin{cases} x + y = 0 \\ z + w - 3y = 0 \end{cases}$$

a occhio! (sapendo che le eq. sono 2)

SECONDO MODO

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in W_{-1} \iff \begin{pmatrix} 0 & -1 & x \\ 0 & 1 & y \\ -1 & 2 & z \\ 1 & 1 & w \end{pmatrix}$$

deve avere 2 pivot $\neq 0$

$$\begin{pmatrix} -1 & 2 & z \\ 0 & -1 & x \\ 0 & 1 & y \\ 0 & 3 & z+w \end{pmatrix} \rightarrow \begin{pmatrix} -1 & 2 & z \\ 0 & -1 & x \\ 0 & 0 & \boxed{x+y} \\ 0 & 0 & \boxed{z+w-3y} \end{pmatrix}$$

$$\begin{cases} x + y = 0 \\ z + w - 3y = 0 \end{cases}$$

Es. 3

$$\begin{cases} a_{n+1} = 5a_n + a_{n-1}, \quad n \geq 1 \\ a_0 = 1 \\ a_1 = 3 \end{cases}$$

Oss. cruciale

$$a_{n+1} = 5a_n + a_{n-1} \equiv a_{n-1} \ (5) \qquad \forall n \geq 1$$

$$a_0 = 1 \qquad a_1 = 3 \qquad a_2 = 16 \qquad a_3 = 83 \quad \dots$$

congrui mod. 5

congrui mod. 5

$$\Rightarrow \begin{cases} a_{2k} \equiv a_0 \equiv 1 \ (5) \\ a_{2k+1} \equiv a_1 \equiv 3 \ (5) \end{cases} \qquad \forall k \geq 0$$

(i)

$$123456789 \equiv -1 \ (5)$$

$\Rightarrow$ non può appartenere ai termini della succ. perché $\not\equiv 1 \ (5)$ e $\not\equiv 3 \ (5)$

(ii) $5 \nmid (a_n + a_{n+1}) \quad \forall n \geq 0$

$\Leftrightarrow \quad a_n + a_{n+1} \not\equiv 0 \ (5) \quad \forall n \geq 0$

Oss.

$a_n + a_{n+1} \equiv 1+3 \ (5) \equiv 4 \ (5)$

perché uno tra n e $n+1$ è pari e l'altro è disp.

$\Rightarrow \not\equiv 0 \ (5)$.

ESERCIZIO

Dimostrare (ii) per induzione.