GEOMETRIA E ALGEBRA LINEARE

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$\boxed{3\ (iii)}$ $z^4 = \bar{z}^3$

- $z = 0$ è sol dell'equazione
- Possiamo supporre $z \neq 0$

$z^4 = \bar{z}^3 \Longrightarrow |z|^4 = |\bar{z}|^3 = |z|^3$

$\xRightarrow{z \neq 0} |z| = 1 \Longrightarrow z = e^{i\theta}$ per qualche θ

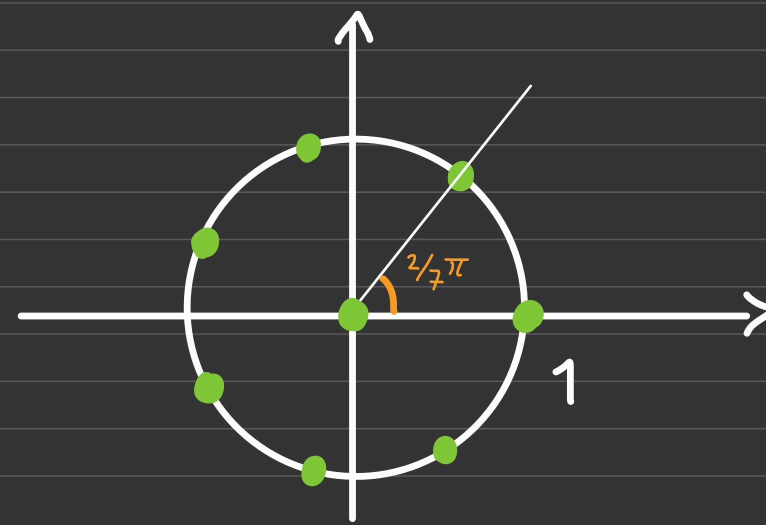
$e^{i4\theta} = e^{-i3\theta} \iff e^{i7\theta} = 1$

$\Leftrightarrow \quad 7\theta = 2k\pi \qquad k = 0, 1, \quad , 6$

$\Leftrightarrow \quad \vartheta = \frac{2}{7}\pi k \qquad k = 0, 1, \quad , 6$

Le sol dell'eq sono $z = 0$ e $z = e^{i\frac{2}{7}\pi k}$

con $k = 0, 1, \quad , 6$



$\boxed{7}$ $U = \{(x,y,z) \in \mathbb{R}^3 \quad x+y-z=0\} \subseteq \mathbb{R}^3$

(1) Base di U

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in U \iff x+y-z=0 \iff z = x+y$$

$$\left.\begin{aligned} &\forall x,y \in \mathbb{R} \quad \begin{pmatrix} x \\ y \\ x+y \end{pmatrix} \in U \\ &= x\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \end{aligned}\right] \Rightarrow U = \operatorname{Span}\left(\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right)$$

$\begin{pmatrix}1\\0\\1\end{pmatrix}$ e $\begin{pmatrix}0\\1\\1\end{pmatrix}$ sono indipendenti?

– Sì perché non sono uno multiplo dell'altro

– $\begin{pmatrix}1 & 0\\0 & 1\\1 & 1\end{pmatrix}$ ha rango 2 $\Rightarrow$ indipendenti

$\Rightarrow \mathcal{B} = \left(\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix} \right)$ è una base di U

$\Rightarrow \dim U = 2$

(ii) $\left[\begin{pmatrix}1\\1\\2\end{pmatrix}\right]_{\mathcal{B}}$

$\mathcal{B}$ base $\Longrightarrow$ $\exists!\, \lambda_1, \lambda_2 \in \mathbb{R}$

$$\begin{pmatrix}1\\1\\2\end{pmatrix} = \lambda_1 \begin{pmatrix}1\\0\\1\end{pmatrix} + \lambda_2 \begin{pmatrix}0\\1\\1\end{pmatrix}$$

$\lambda_1 = \lambda_2 = 1 \Longrightarrow \left[\begin{pmatrix}1\\1\\2\end{pmatrix}\right]_{\mathcal{B}} = \begin{pmatrix}1\\1\end{pmatrix}$

$\boxed{11}$

$$V = \text{Span}\left(\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -3 \\ -3 \\ 3 \end{pmatrix} \right)$$

$$W_t = \text{Span}\left(\begin{pmatrix} -1 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ t+1 \\ 2t+1 \\ 1 \end{pmatrix} \right), \quad t \in \mathbb{R}$$

(1) $\dim V$, $\dim W_t$

I generatori di V sono indipendenti $\Longrightarrow$ $\dim V = 2$

I gen di W_t sono indipendenti $\forall t$ (non sono uno multiplo dell'altro) $\Longrightarrow$ $\dim W_t = 2$ $\forall t \in \mathbb{R}$

(ii) $\dim(V \cap W_t) =$

$$= \dim V + \dim W_t - \dim(V + W_t)$$

per la form di Grassmann

$$V + W_t = \operatorname{Span}\left(\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -3 \\ -3 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ t+1 \\ 2t+1 \\ 1 \end{pmatrix} \right)$$

$$\begin{pmatrix} 1 & -1 & -1 & 0 \\ 1 & -3 & 1 & t+1 \\ 1 & -3 & 2 & 2t+1 \\ 0 & 3 & 1 & 1 \end{pmatrix} \xrightarrow[R_3 - R_1]{R_2 - R_1} \begin{pmatrix} 1 & -1 & -1 & 0 \\ 0 & -2 & 2 & t+1 \\ 0 & -2 & 3 & 2t+1 \\ 0 & 3 & 1 & 1 \end{pmatrix}$$

$$\xrightarrow[R_4 + \frac{3}{2}R_2]{R_3 - R_2} \begin{pmatrix} 1 & -1 & -1 & 0 \\ 0 & -2 & 2 & t+1 \\ 0 & 0 & 1 & t \\ 0 & 0 & 4 & \frac{3}{2}t + \frac{5}{2} \end{pmatrix} \xrightarrow{R_4 - 4R_3} \begin{pmatrix} 1 & -1 & -1 & 0 \\ 0 & -2 & 2 & t+1 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & -\frac{5}{2}t + \frac{5}{2} \end{pmatrix}$$

$t = 1 \Rightarrow 3$ pivot $\neq 0 \Rightarrow$ ci sono 3 colonne indipendenti (le prime 3)

$\Rightarrow \dim(V + W_t) = 3$

$\underset{\text{GRASSMANN}}{\Rightarrow} \dim(V \cap W_t) = 1$

$t \neq 1 \Rightarrow 4$ pivot $\neq 0 \Rightarrow \dim(V + W_t) = 4$

$\Rightarrow \dim(V \cap W_t) = 0 \Rightarrow V \cap W_t = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$

(iii) Eq cartesiane per W_{-1}

$$W_{-1} = \text{Span}\left(\begin{pmatrix} -1 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right)$$

$\dim W_{-1} = 2 \implies$ servono $4-2=2$ eq cartes

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in W_{-1} \iff rk \begin{pmatrix} -1 & 0 & x \\ 1 & 0 & y \\ 2 & -1 & z \\ 1 & 1 & w \end{pmatrix} \leq 2$$

A

B

$\det A = -x - y$

$\det B = -w + 2y + y - z$

$$\Longleftrightarrow \begin{cases} \det A = 0 \\ \det B = 0 \end{cases} \Longleftrightarrow \begin{cases} x + y = 0 \\ 3y - z - w = 0 \end{cases}$$

$$\Downarrow$$

$$\underline{W}_{-1} = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \in \mathbb{R}^4 \quad x + y = 0 \text{ e } 3y - z - w = 0 \right\}$$

$\boxed{18}$ $f \quad \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$$f(x,y,z) = (x-y,\ x+2y+3z,\ y+z)$$

$$A = [f]_{\mathcal{C}}^{\mathcal{C}} = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{pmatrix}$$ matrice di f rispetto alle basi canoniche

(1) Base per Ker f e Im f

$$\operatorname{Im} f = \operatorname{Span}\left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \right)$$

$$A = \begin{pmatrix} 1 & \boxed{\begin{matrix} -1 & 0 \\ 2 & 3 \end{matrix}} \\ 1 & \\ 0 & \begin{matrix} 1 & 1 \end{matrix} \end{pmatrix}$$

$\det A = 2 - 3 + 1 = 0$

$\Rightarrow$ le colonne di A sono dipendenti

$\Rightarrow \operatorname{rk} A = 2 \Rightarrow \dim \operatorname{Im} f = 2$

$$\operatorname{Im} f = \operatorname{Span}\left(\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \right)$$

Una base per $\operatorname{Im} f$ è

$$\left(\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \right)$$

$\dim \operatorname{Ker} f = 3 - \dim \operatorname{Im} f = 1$

dal teorema della dimensione

① $\operatorname{Ker} f = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \quad A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \right\}$

Risolvere il sistema $A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

② $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \in \operatorname{Span}\left(\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \right)$

e $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = -1 \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$

$$\begin{pmatrix}1\\1\\0\end{pmatrix}=f\begin{pmatrix}1\\0\\0\end{pmatrix},\quad \begin{pmatrix}-1\\2\\1\end{pmatrix}=f\begin{pmatrix}0\\1\\0\end{pmatrix},\quad \begin{pmatrix}0\\3\\1\end{pmatrix}=f\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\begin{pmatrix}1\\1\\0\end{pmatrix}= -1\begin{pmatrix}-1\\2\\1\end{pmatrix}+1\begin{pmatrix}0\\3\\1\end{pmatrix} \Longrightarrow$$

$$f(e_1)=-1\ f(e_2)+1\ f(e_3) \Longrightarrow$$

$$f(e_1)+1\ f(e_2)-1\ f(e_3)=0 \overset{f \text{ lineare}}{\Longrightarrow}$$

$$f(e_1+e_2-e_3)=0 \Rightarrow \begin{pmatrix}1\\1\\-1\end{pmatrix}\in \operatorname{Ker} f$$

$$\Rightarrow \left(\begin{pmatrix}1\\1\\-1\end{pmatrix}\right) \text{ è una base di } \operatorname{Ker} f$$

(11) Costruire, se esiste, $g: \mathbb{R}^3 \to \mathbb{R}^3$ lineare tale che $g \neq 0$ e $g \circ f = 0$

$$g \circ f = 0 \iff g(\operatorname{Im} f) = \{0\}$$

Infatti se $w \in \operatorname{Im} f \Rightarrow w = f(v)$ per un qualche $v \in \mathbb{R}^3 \Rightarrow g(w) = g(f(v)) = 0$

$$\mathcal{B} = \Bigg(\underbrace{\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}}_{\text{base di } \operatorname{Im} f}, \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{\text{complet}} \Bigg) \text{ base di } \mathbb{R}^3$$

$$\left.\begin{array}{l} \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \stackrel{g}{\longmapsto} 0 \\ \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \stackrel{g}{\longmapsto} 0 \end{array}\right\} \text{perché vogliamo che } g(\operatorname{Im} f) = \{0\}$$

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \stackrel{g}{\longmapsto} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Big\} \text{ andava bene qualsiasi vett } \neq 0$$

Abbiamo definito un'unica g $\mathbb{R}^3 \to \mathbb{R}^3$ lineare perché abbiamo definito g su una base di $\mathbb{R}^3$ (ossia $\mathcal{B}$)

Inoltre $g \neq 0$ perché $g\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix} \neq 0$

e $g(\mathrm{Im} f) = \{0\}$ perché g manda in 0 i generatori di $\mathrm{Im} f$, e quindi tutto $\mathrm{Im} f$ per linearità

DOMANDA

$$[g]^{\mathcal{B}}_{\mathcal{C}} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad [g]^{\mathcal{C}}_{\mathcal{E}} = ?$$