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3 (iv) $z^2 = -i|z| - \sqrt{2}$

$(x+iy)^2 = -i\sqrt{x^2+y^2} - \sqrt{2}$

$x^2 - y^2 + 2xyi = -i\sqrt{x^2+y^2} - \sqrt{2}$

$$\iff \begin{cases} x^2 - y^2 = -\sqrt{2} \\ 2xy = -\sqrt{x^2+y^2} \end{cases}$$

finire per casa

COORDINATE

V sp vettoriale su $\mathbb{K}$, $\mathcal{B} = (v_1, \quad , v_n)$

$\Rightarrow \forall v \in V \; \exists ! \; \lambda_1, \quad , \lambda_n \in \mathbb{K} \qquad v = \sum_{i=1}^{n} \lambda_i v_i$

coordinate di v risp a $\mathcal{B}$

Definiamo

$$\phi_{\mathcal{B}} \quad V \to \mathbb{K}^n, \quad \phi_{\mathcal{B}}(v) = \begin{pmatrix} \lambda_1 \\ \\ \lambda_n \end{pmatrix}$$

Teorema $\phi_{\mathcal{B}}$ è un isomorfismo

NOTAZIONE $\phi_{\mathcal{B}}(v) = [v]_{\mathcal{B}}$

Oss $V = \mathbb{K}^n$, $\mathcal{B} = \mathcal{C} \Rightarrow [v]_{\mathcal{C}} = v$

$(\phi_{\mathcal{C}} = id_{\mathbb{K}^n})$

MATRICE DI UN'APP LINEARE

① $A \in M(m,n,\mathbb{K})$ matrice

$f_A \quad \mathbb{K}^n \to \mathbb{K}^m, \quad f_A(x) = Ax$

$\Rightarrow f_A$ è lineare $\left(f_A \in \mathcal{L}(\mathbb{K}^n, \mathbb{K}^m) \right)$

② $$\begin{aligned} M(m,n,\mathbb{K}) &\xrightarrow{\Phi} \mathcal{L}(\mathbb{K}^n,\mathbb{K}^m) \\ A &\mapsto f_A \end{aligned}$$

- Φ è isomorfismo

- $A \in M(m,n,\mathbb{K})$, $f_A \quad \mathbb{K}^n \to \mathbb{K}^m$

$\Rightarrow f_A(e_i) = Ae_i = A^i$ ← i-esima colonna di A

$i = 1, \quad , n$

- Data $f \in \mathcal{L}(\mathbb{K}^n, \mathbb{K}^m)$, se consideriamo

$A = \left(f(e_1) \,|\, f(e_2) \,|\, \quad | \, f(e_n) \right) \in M(m,n,\mathbb{K})$

$\Rightarrow f = f_A = \Phi(A)$

③ V, W sp vett $\quad \dim V = n, \ \dim W = m$

$\mathcal{B}_V = (v_1, \ \ , v_n) \quad \mathcal{B}_W = (w_1, \ \ , w_m)$

$f \ V \to W$ lineare

$$\begin{array}{ccc} V & \xrightarrow{\ f\ } & W \\ \phi_{\mathcal{B}_V} \downarrow & & \downarrow \phi_{\mathcal{B}_W} \\ \mathbb{K}^n & \xrightarrow{\ g\ } & \mathbb{K}^m \end{array}$$

$g = \phi_{\mathcal{B}_W} \circ f \circ \phi_{\mathcal{B}_V}^{-1}$

② $\Longrightarrow$ g è rapp da una matrice $A_g \in M(m, n, \mathbb{K})$

Def $M^{B_V}_{B_W}(f) = [f]^{B_V}_{B_W} = A_g$

$$\left([f]^{B_V}_{B_W}\right)^i = [f]^{B_V}_{B_W} e_i = g(e_i) =$$

$$= \phi_{B_W}\left(f\left(\phi^{-1}_{B_V}(e_i)\right)\right) =$$

$$= \phi_{B_W}\left(f(v_i)\right) \quad i = 1, \ldots, n$$

La i-esima col di $[f]^{B_V}_{B_W}$ sono le coord risp a B_W dell'immagine dell'i-esimo vettore di B_V

$$[f]^{\mathcal{B}_V}_{\mathcal{B}_W} = \left(\begin{array}{c|c} \; [f(v_1)]_{\mathcal{B}_W} \; & \; \cdots \; \end{array} \right)$$

④ $f : V \to W$, $g : W \to Z$ lineari

$\mathcal{B}_V, \mathcal{B}_W, \mathcal{B}_Z$ basi

$$\Rightarrow [g \cdot f]^{\mathcal{B}_V}_{\mathcal{B}_Z} = [g]^{\mathcal{B}_W}_{\mathcal{B}_Z} \, [f]^{\mathcal{B}_V}_{\mathcal{B}_W}$$

(5) $f: V \to V$, $\mathcal{B}$ e $\mathcal{B}'$ basi di V

$$\begin{array}{ccc} V_{\mathcal{B}} & \xrightarrow{\ f\ } & V_{\mathcal{B}} \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle id} \\ V_{\mathcal{B}'} & \xrightarrow[\ f\ ]{} & V_{\mathcal{B}'} \end{array}$$

$$f = id \circ f \circ id$$

(4)

$$\Rightarrow [f]^{\mathcal{B}'}_{\mathcal{B}'} = [id]^{\mathcal{B}}_{\mathcal{B}'} \, [f]^{\mathcal{B}}_{\mathcal{B}} \, [id]^{\mathcal{B}'}_{\mathcal{B}}$$

matrice di camb di base da $\mathcal{B}$ a $\mathcal{B}'$

$$V_{\mathcal{B}} \underset{id}{\overset{id}{\rightleftarrows}} V_{\mathcal{B}'} \Rightarrow [id]^{\mathcal{B}}_{\mathcal{B}'} = \left([id]^{\mathcal{B}'}_{\mathcal{B}} \right)^{-1}$$

<u>Oss</u> $\mathbb{K}^n$, $\mathcal{B} = (v_1, \quad , v_n)$ base

$id \quad \mathbb{K}^n \to \mathbb{K}^n$

$$\Rightarrow [id]^{\mathcal{B}}_{\mathcal{C}} = \left(v_1 \mid v_2 \mid \quad \mid v_n \right)$$

19 $f: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad f\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ e $f\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

(i) Perché $\left(\begin{pmatrix} 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}\right)$ è una base di $\mathbb{R}^2$

(ii) $\mathcal{B}' = \left(\begin{pmatrix} 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}\right) \Longrightarrow [f]^{\mathcal{B}'}_{\mathcal{C}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\begin{array}{ccc} \mathbb{R}^2_{\mathcal{B}'} & \xrightarrow{\ f\ } & \mathbb{R}^2_{\mathcal{C}} \\ \mathrm{id} \downarrow & & \downarrow \mathrm{id} \\ \mathbb{R}^2_{\mathcal{C}} & \xrightarrow{\ f\ } & \mathbb{R}^2_{\mathcal{C}} \end{array} \Longrightarrow [f]^{\mathcal{C}}_{\mathcal{C}} = [\mathrm{id}]^{\mathcal{C}}_{\mathcal{C}} [f]^{\mathcal{B}'}_{\mathcal{C}} [\mathrm{id}]^{\mathcal{C}}_{\mathcal{B}'},$$

$$[\mathrm{id}]^{\mathcal{C}}_{\mathcal{C}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$[id]^{C}_{B'} = \left([id]^{B'}_{C}\right)^{-1} = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}$$

$$[f]^{C}_{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix}$$

(iii) $[f]^{B}_{B'}$ $\qquad B = \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$

$$\begin{array}{ccc} \mathbb{R}^2_{C} & \xrightarrow{f} & \mathbb{R}^2_{C} \\ {\scriptstyle id}\downarrow & & \downarrow{\scriptstyle id} \\ \mathbb{R}^2_{B} & \xrightarrow{f} & \mathbb{R}^2_{B'} \end{array}$$

$$[f]^{B}_{B'} = [id]^{C}_{B'} [f]^{C}_{C} [id]^{B}_{C} =$$

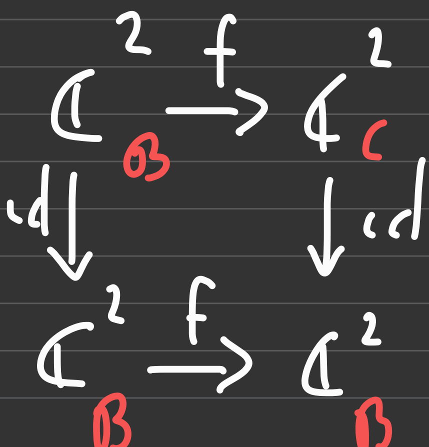
$$= \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -5 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 9 & -5 \\ -16 & 9 \end{pmatrix}$$

20 $f: \mathbb{C}^2 \to \mathbb{C}^2 \qquad f(x,y) = (-y, 2x)$

$\mathcal{B} = \left(\begin{pmatrix} i \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)$

$[f]^{\mathcal{B}}_{\mathcal{B}} = ? \qquad f\begin{pmatrix} i \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2i \end{pmatrix}, \quad f\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

$\Longrightarrow [f]^{\mathcal{B}}_{\mathcal{C}} = \begin{pmatrix} 1 & 0 \\ 2i & 2 \end{pmatrix}$



$\Longrightarrow [f]^{\mathcal{B}}_{\mathcal{B}} = [id]^{\mathcal{C}}_{\mathcal{B}} \, [f]^{\mathcal{B}}_{\mathcal{C}} =$

$= \begin{pmatrix} i & 1 \\ -1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 2i & 2 \end{pmatrix} = \begin{pmatrix} -2i & -2 \\ -1 & 2i \end{pmatrix}$

Oss $f(x,y) = (-y, 2x) = \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$\Rightarrow [f]_C^C = \begin{pmatrix} 0 & -1 \\ 2 & 0 \end{pmatrix}$

La matrice A che definisce f per moltiplicazione è anche $[f]_C^C$, poiché

$$([f]_C^C)^i = \phi_C(f(e_i)) = f(e_i) =$$

$$= A\, e_i = A^i \qquad i = 1, \ldots, n$$

23

$$A_h = \begin{pmatrix} h+1 & 2 & h+5 & 0 \\ h & 1 & 2 & h \\ h & h & 3h+1 & -1 \end{pmatrix} \in M(3,4)$$

(B = submatrix of columns 1, 2, 4)

$$f_h \; \mathbb{R}^4 \to \mathbb{R}^3, \quad f_h(X) = A_h X$$

(1) $\dim \operatorname{Im} f_h = ?$

$$\det B = -h - 1 + 2h^2 - h^3 - h^2 + 2h =$$

$$= -h^3 + h^2 + h - 1 = -h^2(h-1) + h - 1 =$$

$$= (h-1)(1-h^2) = -(1+h)(1-h)^2$$

- $h \neq \pm 1 \Rightarrow \dim \operatorname{Im} f_h = 3$
- $h = 1$

$$A_1 = \begin{pmatrix} 2 & 2 & 6 & 0 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 4 & -1 \end{pmatrix}$$

M (the boxed submatrix $\begin{pmatrix} 2 & 6 & 0 \\ 1 & 2 & 1 \\ 1 & 4 & -1 \end{pmatrix}$)

$$3M^1 - M^2 = M^3 \Rightarrow \det M = 0$$

$$\Rightarrow \operatorname{rk} A_1 \leq 2 \quad \text{ma} \quad \det \begin{pmatrix} 6 & 0 \\ 2 & 1 \end{pmatrix} \neq 0$$

$$\Rightarrow \operatorname{rk} A_1 = 2 \Rightarrow \dim \operatorname{Im} A_1 = 2$$

• $h = -1$ $\quad A_{-1} = \begin{pmatrix} 0 & 2 & 4 & 0 \\ -1 & 1 & 2 & -1 \\ -1 & -1 & -2 & -1 \end{pmatrix}$

(matrice evidenziata nelle prime tre colonne: C)

$C^3 = 2C^2 \Rightarrow \det C = 0 \Rightarrow \dim \operatorname{Im} A_{-1} = 2$

(ii) Per quali $h \in \mathbb{R}$ $\operatorname{Ker} f_h \cong \operatorname{Im} f_h$?

$$\dim \operatorname{Ker} f_h = \begin{cases} 4-3 = 1 & h \neq \pm 1 \\ 4-2 = 2 & h = 1 \vee h = -1 \end{cases}$$

$\operatorname{Ker} f_h \cong \operatorname{Im} f_h \Leftrightarrow$ hanno la stessa dimensione $\Leftrightarrow$ $h = 1 \vee h = -1$

(iii) $h=1$ (f_h non surgettiva)

(ii) $\Rightarrow \operatorname{Im} f_1 = \left\langle \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\rangle \subseteq \mathbb{R}^3$

Ci serve 1 equazione per def $\operatorname{Im} f_h$

$y+z-x=0$ è soddisfatta da $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ e $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \operatorname{Im} f_1 \Leftrightarrow rk \begin{pmatrix} 2 & 0 & x \\ 1 & 1 & y \\ 1 & -1 & z \end{pmatrix} = 2$

$\Leftrightarrow 0 = \det \begin{pmatrix} 2 & 0 & x \\ 1 & 1 & y \\ 1 & -1 & z \end{pmatrix} = y+z-x$

$\Longrightarrow \operatorname{Im} f_1 = \{ y+z-x=0 \}$

(iv) $H = \{x+y=0,\ z=0\} \subseteq \mathbb{R}^4$

Per quali $h \in \mathbb{R}$ $\mathbb{R}^4 = H \oplus \operatorname{Ker} f_h$?

- $\dim H = 2$

- $\mathbb{R}^4 = H \oplus \operatorname{Ker} f_h \implies \dim \operatorname{Ker} f_h = 2$

(il viceversa NON vale)

$\implies$ bisogna controllare per $h = 1$ o $h = -1$

$$h=1 \Rightarrow A_1 = \begin{pmatrix} 2 & 2 & 6 & 0 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 4 & -1 \end{pmatrix}$$

$$\quad w_1 \quad w_2 \quad w_3 \quad w_4$$

$$0 = w_1 - w_2 = f_h(e_1) - f_h(e_2) = f_h(e_1 - e_2)$$

$$\Rightarrow e_1 - e_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \in \ker f_h$$

$$0 = 3w_2 - w_3 - w_4 = f_h\begin{pmatrix} 0 \\ 3 \\ -1 \\ -1 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 \\ 3 \\ -1 \\ -1 \end{pmatrix} \in \ker f_h$$

$$\mathrm{Ker} f_h = \left\langle \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ -1 \\ -1 \end{pmatrix} \right\rangle = \left\{ \begin{pmatrix} a \\ -a+3b \\ -b \\ -b \end{pmatrix} \quad a, b \in \mathbb{R} \right\}$$

$$\mathrm{Ker} f_h \cap H$$

$$\begin{pmatrix} a \\ -a+3b \\ -b \\ -b \end{pmatrix} \in H \Leftrightarrow \begin{cases} a + (-a+3b) = 0 \\ b = 0 \end{cases} \Leftrightarrow b = 0$$

$$\Rightarrow \mathrm{Ker} f_h \cap H = \left\{ \begin{pmatrix} a \\ -a \\ 0 \\ 0 \end{pmatrix} \quad a \in \mathbb{R} \right\} = \left\langle \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$$

$$\Rightarrow \mathbb{R}^4 \neq H \oplus \mathrm{Ker} f_h$$

$$h = -1 \Rightarrow A_{-1} = \begin{pmatrix} 0 & 2 & 4 & 0 \\ -1 & 1 & 2 & -1 \\ -1 & -1 & -2 & -1 \end{pmatrix}$$

$$\ker f_h = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -1 \\ 0 \end{pmatrix} \right\rangle$$

$$\ker f_h \cap H = \{0\} \Rightarrow \mathbb{R}^4 = H \oplus \ker f_h$$