$\boxed{1}$ $p(z) = z^3 - z^2 + z + 1 + a \in \mathbb{R}[z]$

(a) $z = -i$ è radice di p $\iff p(-i) = 0$

$$\cancel{i} + 1 - \cancel{i} + 1 + a = 0$$

$$a = -2$$

(b) $z^3 - z^2 + z - 1 = z^2(z-1) + z - 1 =$

$$= (z-1)(z^2+1)$$

$\Longrightarrow$ radici $1, i, -i$

$\boxed{2}$ $\begin{cases} kx + z = 1 \\ x + ky + z = 2 \\ -x + y - z = k-1 \end{cases}$ $k \in \mathbb{R}$

$$A = \begin{pmatrix} k & 0 & 1 \\ 1 & k & 1 \\ -1 & 1 & -1 \end{pmatrix}$$ matrice del sistema

$$\det A = -k^2 + 1 + \cancel{k} - \cancel{k} = 1 - k^2$$

$$\Rightarrow \quad \operatorname{rk} A = \begin{cases} 3 & \text{se } k \neq 1 \text{ e } k \neq -1 \\ 2 & \text{se } k = 1 \text{ o } k = -1 \end{cases}$$

Per $k \neq \pm 1$ c'è un'unica sol

Per $k = 1$

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ -1 & 1 & -1 \end{pmatrix} \qquad A|b = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ -1 & 1 & -1 & 0 \end{pmatrix}$$

$rk\,A = 2$ e $rk\,A|b = 2 \Longrightarrow$

è risolubile perché ha matrice $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$
e quindi ha infinite sol

PER CASA fare $k = -1$, che è analogo

$\boxed{3}$ $V = \mathbb{R}_{\leq 2}[x]$, $W = \langle 1 - x + x^2, 2 + 2x - x^2 \rangle$

Per quali K $12 + 8x + Kx^2 \in W$?

Scegliendo $\mathcal{B}_V = (1, x, x^2)$, $V \cong_{\mathcal{B}_V} \mathbb{R}^3$
e ogni polinomio è rapp dal suo vettore dei coefficienti

$$\Longrightarrow \quad \tilde{W} = \left\langle \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix} \right\rangle$$

Per quali K $\begin{pmatrix} 12 \\ 8 \\ K \end{pmatrix} \in \tilde{W}$?

$$A_K = \begin{pmatrix} 1 & 2 & 12 \\ -1 & 2 & 8 \\ 1 & -1 & K \end{pmatrix}$$

$$\begin{pmatrix} 12 \\ 8 \\ K \end{pmatrix} \in \tilde{W} \iff \operatorname{rk} A_K < 3 \iff \det A_K = 0$$

$$\det A_K = 2K + 16 + 12 - 24 + 8 + 2K =$$

$$= 4K + 12$$

$$\det A_K = 0 \iff K = -3$$

$\boxed{4}$ $f_h \quad \mathbb{R}^3 \to \mathbb{R}^3$

$$f_h(x,y,z) = (x, 0, x + hy + h^2 z)$$

$$\Rightarrow A_h = [f_h]_C^C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & h & h^2 \end{pmatrix}$$

(a) f_h surgettiva $\Leftrightarrow$ $\mathrm{rk}\, A_h = 3$

ma A_h ha una riga di 0

$\Rightarrow \mathrm{rk}\, A_h < 3 \quad \forall h \in \mathbb{R}$

$\Rightarrow f_h$ mai surgettiva

(b) $h=3 \Rightarrow A_h = \begin{array}{c} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 3 & 9 \end{pmatrix} \\ \begin{array}{ccc} w_1 & w_2 & w_3 \end{array} \end{array}$

$\Rightarrow \dim \operatorname{Ker} A_h = 3 - \dim \operatorname{Im} A_h =$

$= 3 - 2 = 1$

<u>BASE DI $\operatorname{Ker} A_h$</u> $(h=3)$

$3e_2 - e_3$

$$0 = 3w_2 - w_3 = 3f_h(e_2) - f_h(e_3) = f_h\begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix}$$

$$\Rightarrow \operatorname{Ker} f_h = \left\langle \begin{pmatrix} 0 \\ 3 \\ -1 \end{pmatrix} \right\rangle$$

(c) Per quali $h \in \mathbb{R}$ $\mathbb{R}^3 = \operatorname{Ker} f_h \oplus \operatorname{Im} f_h$?

$h = 0 \Rightarrow \dim \operatorname{Im} f_h = 1$

$h \neq 0 \Rightarrow \dim \operatorname{Im} f_h = 2$

$\boxed{h = 0}$ $\operatorname{Im} f_h = \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\rangle$ $\operatorname{Ker} f_h = \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle$

$\Rightarrow \operatorname{Im} f_h \cap \operatorname{Ker} f_h = \{0\}$

GRASSMANN

$\Rightarrow \dim(\operatorname{Ker} f_h + \operatorname{Im} f_h) = 2 + 1 = 3$

$\Rightarrow \mathbb{R}^3 = \operatorname{Ker} f_h \oplus \operatorname{Im} f_h$

$\boxed{h \neq 0}$

$$A_h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & h & h^2 \end{pmatrix}$$

$$\operatorname{Im} f_h = \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix} \right\rangle$$

$$0 = h\, w_2 - w_3 = f_h \begin{pmatrix} 0 \\ h \\ -1 \end{pmatrix} \Rightarrow \operatorname{Ker} f_h = \left\langle \begin{pmatrix} 0 \\ h \\ -1 \end{pmatrix} \right\rangle$$

$$\Rightarrow \operatorname{Ker} f_h \cap \operatorname{Im} f_h = \{0\}$$

$$\Rightarrow \dim(\operatorname{Ker} f_h + \operatorname{Im} f_h) = 1 + 2 = 3$$

$$\Rightarrow \mathbb{R}^3 = \operatorname{Ker} f_h \oplus \operatorname{Im} f_h$$

(d) $\quad \mathcal{B} = \left(\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right)$

$M_h = [f_h]_C^B$ per $h = 3$

$$(M_h)^1 = f_h \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 10 \end{pmatrix}$$

$$(M_h)^2 = f_h \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 13 \end{pmatrix} \implies M_h = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 10 & 13 & 1 \end{pmatrix}$$

$$(M_h)^3 = f_h \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

(e) $h = 1$ $\quad L = \{ g \ \mathbb{R}^3 \to \mathbb{R}^3 \ \ g \cdot f_h = 0 \}$

$$g \cdot f_h = 0 \iff g(\mathrm{Im} f_h) = \{0\}$$

$$\mathrm{Im} f_h = \langle \underbrace{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}_{w_1}, \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{w_2} \rangle$$

Scegliamo $\mathcal{B} = (w_1, w_2, v)$, con v che completa (w_1, w_2) a base di $\mathbb{R}^3$

Data $g \in L$, consideriamo $[g]^{\mathcal{B}}_{\mathcal{C}}$

$$[g]^{\mathcal{B}}_{\mathcal{C}} = \left(\begin{array}{cc|c} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & * \end{array}\right) \in M(3,3,\mathbb{R})$$

$\Rightarrow$ Possiamo "vedere" L come

$$\tilde{L} = \{ M \in M(3,3,\mathbb{R}) \quad M^1 = M^2 = 0 \}$$

(cioè $L \cong \tilde{L}$ tramite $g \mapsto [g]^{\mathcal{B}}_{\mathcal{C}}$)

$$\Rightarrow \dim L = \dim \tilde{L} = 3$$

PROIEZIONI SU SOTTOSPAZI

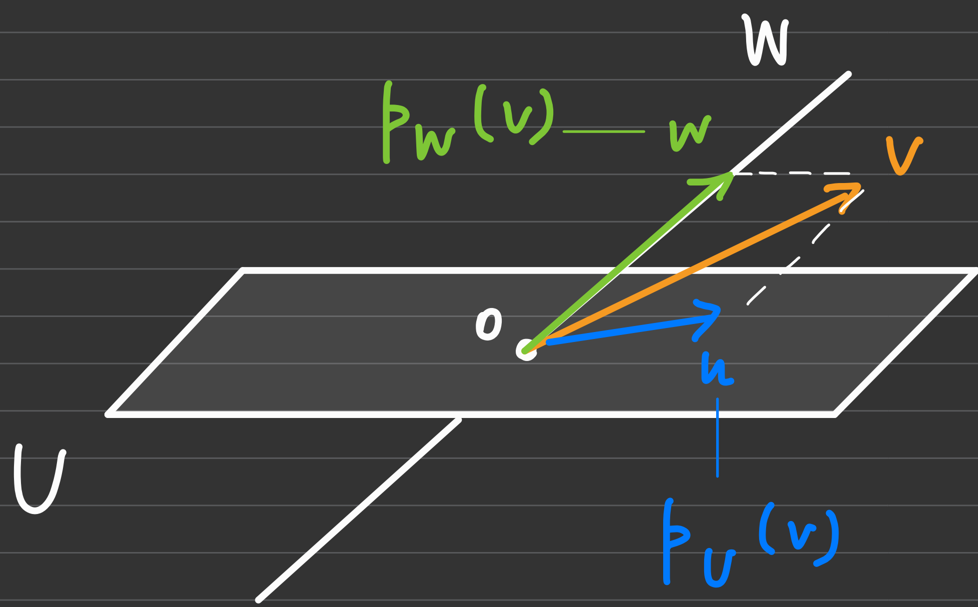
V sp vett $\quad V = U \oplus W$

$\Rightarrow \forall v \in V \ \exists!\, u \in U, w \in W \quad v = u + w$

Def $p_U \quad V \to V, \quad p_U(v) = u$, dove u è l'unico el di U t.c. $v = u + w$ con $w \in W$

Prop p_U è lineare

Idea geom $V = \mathbb{R}^3$, $\dim U = 2$, $\dim W = 1$



Prop (i) $p_U|_U = id_U$ ($p_U(u) = u \quad \forall u \in U$)

(ii) $\mathrm{Im}\, p_U = U$, $\ker p_U = W$

($\Rightarrow$ $\ker p_U \oplus \mathrm{Im}\, p_U = U \oplus W = V$)

(iii) $p_U^2 = p_U$ (idempotente)

(iv) $p_U \circ p_W = p_W \circ p_U = 0$

PER CASA dare la prova delle proprietà

Prop $n = \dim V$, $d_U = \dim U$, $d_W = \dim W$

$\mathcal{B} = (\mathcal{B}_U, \mathcal{B}_W)$ $\Rightarrow$ $[p_U]^{\mathcal{B}}_{\mathcal{B}} = \left(\begin{array}{c|c} I_{d_U} & 0 \\ \hline 0 & O_{d_W} \end{array}\right)$

Teorema $f: V \to V$ lineare, $f^2 = f$

$\Rightarrow \exists\, U \leq V$ sottosp. $f = p_U$

[Sugg
- $V = \operatorname{Ker} f \oplus \operatorname{Im} f$
- $U = \operatorname{Im} f$ e mostrare che $f = p_U$]

$\boxed{1}$ $\quad \mathbb{R}^3 = U \oplus W \qquad U = \left\langle \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle$

$$W = \{ y = 0 \text{ e } z = 0 \} = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$$

$$[p_U]_C^C = ?$$

$$\mathcal{B} = \left(\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right) \Longrightarrow [p_U]_{\mathcal{B}}^{\mathcal{B}} = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array} \right)$$

$$\begin{tikzcd} \mathbb{R}^3_{\mathcal{B}} \arrow[r, "p_U"] \arrow[d, "id"'] & \mathbb{R}^3_{\mathcal{B}} \arrow[d, "id"] \\ \mathbb{R}^3_C \arrow[r, "p_U"] & \mathbb{R}^3_C \end{tikzcd}$$

$$[p_U]_C^C = [id]_C^{\mathcal{B}} \, [p_U]_{\mathcal{B}}^{\mathcal{B}} \left([id]_C^{\mathcal{B}} \right)^{-1}$$

$\boxed{2}$ $V = \mathbb{R}_{\leq 3}[x]$

$U = \{ p \in V \quad p(0) = p(1) = 0 \}$

$W = \langle x^2 + 3, x^2 - 3 \rangle$

(a) Provare che $V = U \oplus W$

(b) $[p_U]^{e}_{e}$ $\quad (e = (1, x, x^2, x^3))$

PER CASA svolgere