$\boxed{2}$ $V = \mathbb{R}_{\leq 3}[x]$

$U = \{ p \in V \quad p(0) = p(1) = 0 \}$

$W = \langle x^2 + 3 , \; x^2 - 3 \rangle$

(a) Provare che $V = U \oplus W$

$V \cong \mathbb{R}^4$

$$W \cong \left\langle \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\rangle \Rightarrow \dim W = 2$$

$$p(x) = \sum_{i=0}^{3} a_i x^i$$

$$p(0) = 0 \iff a_0 = 0$$

$$p(1) = 0 \iff \cancel{a_0} + a_1 + a_2 + a_3 = 0$$

$$U \cong \{ (a_0, a_1, a_2, a_3) \mid a_0 = 0, \ a_1 + a_2 + a_3 = 0 \}$$

$$\Rightarrow \dim U = 4 - 2 = 2$$

$$U = \left\langle \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right\rangle$$

$U \cap W$

$$\lambda \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} \in U \iff \begin{cases} 3\lambda - 3\mu = 0 \\ \lambda + \mu = 0 \end{cases}$$

$$\iff \lambda = \mu = 0$$

$$\Rightarrow \quad U \cap W = \{0\} \Rightarrow \dim(U+W) = 4$$

$$\Rightarrow U + W = V$$

e la somma è diretta

(b) $[p_U]_C^C$

$$\mathcal{B} = \Bigg(\underbrace{\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}}_{\mathcal{B}_U}, \underbrace{\begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{\mathcal{B}_W} \Bigg)$$

$$\Longrightarrow [p_U]_{\mathcal{B}}^{\mathcal{B}} = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

e si fa un cambio di base ($\mathcal{B} \to C$)

$\boxed{1}$ $a > 0$, $z^3 = a\bar{z}^2$

- $z = 0$ è sol
- $z \neq 0 \Rightarrow z = \rho e^{i\theta}$

$$\rho^{\cancel{3}} e^{i3\theta} = a \cancel{\rho^2} e^{-i2\theta}$$

$$\rho = a \quad \wedge \quad e^{i3\theta} = e^{-i2\theta}$$

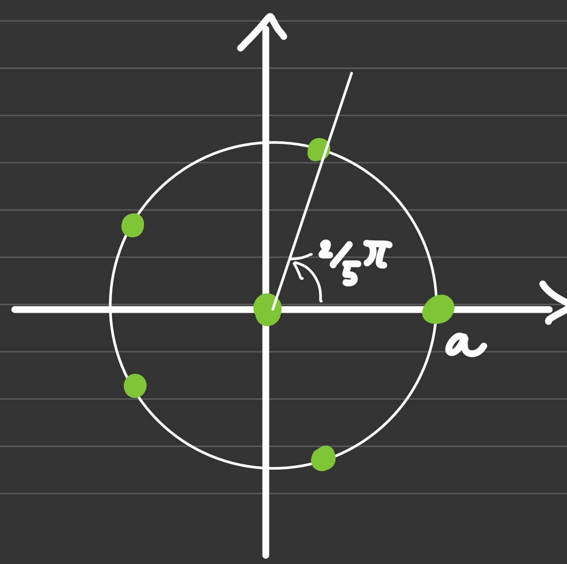
$$e^{i5\theta} = 1$$

$\Rightarrow$ 6 soluzioni

$$e^{i5\theta} = 1 = e^{i0}$$

$$5\theta = 0 + 2k\pi \quad (k \in \mathbb{Z})$$

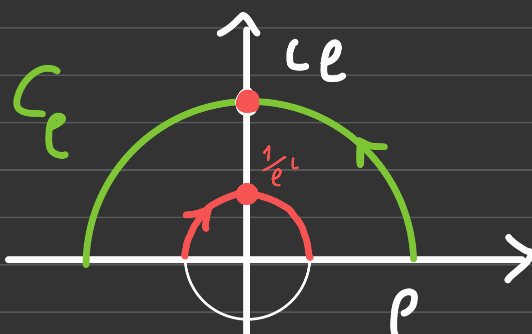
$$\theta = \frac{2}{5}k\pi \quad (k = 0, 1, \quad , 4)$$



$\boxed{2}$ $f \quad \mathbb{C} \setminus \{0\} \to \mathbb{C}$, $f(z) = -\frac{1}{z}$

$C_e = \{ z \in \mathbb{C} \quad |z| = e \, , \, 0 \leq \arg(z) \leq \pi \}$

$|f(z)| = \frac{1}{|z|}$ $\quad |z| = e \Rightarrow |f(z)| = \frac{1}{e}$



$f(ie) = -\frac{1}{ie} = \frac{1}{e} i$

$\Rightarrow f(C_e)$ non può essere la semicirc inferiore

$$z \in C_\rho \iff z = \rho e^{i\theta} \quad 0 \leqslant \theta \leqslant \pi$$

$$f(z) = -\frac{1}{\rho e^{i\theta}} = -\frac{1}{\rho} e^{-i\theta} = \frac{1}{\rho} \underbrace{e^{i\pi}}_{-1} e^{-i\theta} =$$

$$= \frac{1}{\rho} e^{i(\pi-\theta)}$$

$$[0, \pi]$$

$$\Downarrow$$

$$\implies |f(z)| = \frac{1}{\rho}, \quad \arg f(z) = \pi - \theta$$

$\boxed{3}$ $R^3 = U \oplus W$

$$U = \{x + 2y = 0\}, \quad W = \left\langle \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right\rangle$$

$$p_U \begin{pmatrix} -9 \\ 3 \\ 8 \end{pmatrix} = ?$$

$$\begin{pmatrix} -9 \\ 3 \\ 8 \end{pmatrix} = u + \lambda \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \text{ con } u \in U \text{ e } \lambda \in \mathbb{R}$$

$$\Leftrightarrow \begin{pmatrix} -9 \\ 3 \\ 8 \end{pmatrix} - \lambda \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \in U \Leftrightarrow -9 - \lambda + 2 \cdot 3 = 0$$

$$\lambda = -3$$

$$\Rightarrow p_U\begin{pmatrix}-9\\3\\8\end{pmatrix}=\begin{pmatrix}-9\\3\\8\end{pmatrix}-(-3)\begin{pmatrix}1\\0\\2\end{pmatrix}=\begin{pmatrix}-6\\3\\14\end{pmatrix}$$

MODO "CLASSICO"

$$U=\left\langle\begin{pmatrix}-2\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\rangle$$

$$\begin{pmatrix}-9\\3\\8\end{pmatrix}=\overbrace{a\begin{pmatrix}-2\\1\\0\end{pmatrix}+b\begin{pmatrix}0\\0\\1\end{pmatrix}}^{p_U\begin{pmatrix}-9\\3\\8\end{pmatrix}}+c\begin{pmatrix}1\\0\\2\end{pmatrix}$$

$$\Leftrightarrow\begin{cases}-2a+c=-9\\a=3\\b+2c=8\end{cases}\Leftrightarrow\begin{cases}a=3\\b=14\\c=-3\end{cases}$$

$$\Rightarrow p_U\begin{pmatrix}-9\\3\\8\end{pmatrix}=$$

$$=3\begin{pmatrix}-2\\1\\0\end{pmatrix}+14\begin{pmatrix}0\\0\\1\end{pmatrix}=$$

$$=\begin{pmatrix}-6\\3\\14\end{pmatrix}$$

Oss $f \; V \to V \qquad \mathcal{B}, \mathcal{B}'$ basi di V

$$\Rightarrow [f]^{\mathcal{B}'}_{\mathcal{B}'} = [id]^{\mathcal{B}}_{\mathcal{B}'} \, [f]^{\mathcal{B}}_{\mathcal{B}} \left([id]^{\mathcal{B}}_{\mathcal{B}'}\right)^{-1}$$

$$\Rightarrow \det [f]^{\mathcal{B}'}_{\mathcal{B}'} = \det [id]^{\mathcal{B}}_{\mathcal{B}'} \det [f]^{\mathcal{B}}_{\mathcal{B}} \frac{1}{\det [id]^{\mathcal{B}}_{\mathcal{B}'}} =$$

$$= \det [f]^{\mathcal{B}}_{\mathcal{B}}$$

Il det di $[f]^{\mathcal{B}}_{\mathcal{B}}$ <u>NON</u> dipende da $\mathcal{B}$

$$\Rightarrow \det [p_U]^{\mathcal{B}}_{\mathcal{B}} = 0 \quad (\mathcal{B} = \mathcal{B}_U \cup \mathcal{B}_W)$$

$\boxed{5}$ $f_K \quad \mathbb{R}^3 \longrightarrow \mathbb{R}^4$

$$A_K = \begin{pmatrix} K & 0 & K \\ 2K & 1 & 2K-1 \\ K & -1 & K^2-1 \\ 0 & -1 & 1 \end{pmatrix} \qquad v_\alpha = \begin{pmatrix} 1 \\ 3 \\ \alpha-2 \\ \alpha-3 \end{pmatrix}$$

(a) Per quali $K \in \mathbb{R}$ $\quad v_\alpha \in \operatorname{Im} f_K \quad \forall \alpha$?

$$\begin{pmatrix} K & 0 & K & 1 \\ 2K & 1 & 2K-1 & 3 \\ K & -1 & K^2-1 & \alpha-2 \\ 0 & -1 & 1 & \alpha-3 \end{pmatrix} \longrightarrow \begin{pmatrix} K & 0 & K & 1 \\ 0 & 1 & -1 & 1 \\ 0 & -1 & K^2-K-1 & \alpha-3 \\ 0 & -1 & 1 & \alpha-3 \end{pmatrix}$$

$$\begin{pmatrix} k & 0 & k & 1 \\ 0 & 1 & -1 & 1 \\ 0 & -1 & k^2-k-1 & \alpha-3 \\ 0 & -1 & 1 & \alpha-3 \end{pmatrix} \to \left(\begin{array}{ccc|c} k & 0 & k & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & k^2-k-2 & \alpha-2 \\ 0 & 0 & 0 & \alpha-2 \end{array}\right)$$

$$k^2-k-2=0 \iff k=-1 \vee k=2$$

- $k \neq 0,\ k \neq -1,\ k \neq 2 \Rightarrow rk\, A_k = 3$

  Se $\alpha \neq 2 \Rightarrow v_\alpha \notin \operatorname{Im} f_k$

  Se $\alpha = 2 \Rightarrow v_\alpha \in \operatorname{Im} f_k$

• $K = 0$

$$\left(\begin{array}{c|cc|c} \cancel{0} & 0 & 0 & 1 \\ \cancel{0} & 1 & -1 & 1 \\ \cancel{0} & 0 & -2 & \alpha-2 \\ \cancel{0} & 0 & 0 & \alpha-2 \end{array}\right)$$

M

$\operatorname{rk} M = 3 \quad \forall \alpha$

$\Updownarrow$

$v_\alpha \notin \operatorname{Im} f_K \quad \forall \alpha$

• $K = -1$, $K = 2$ analogo

$\Longrightarrow$ per nessun $K \in \mathbb{R}$ succede che $v_\alpha \in \operatorname{Im} f_K \quad \forall \alpha$