$\boxed{11}$ $A \in \mathrm{Mat}(2,\mathbb{R})$

$f_A \quad \mathrm{Mat}(2,\mathbb{R}) \to \mathrm{Mat}(2,\mathbb{R}), \quad f_A(X) = AX$

(i) $\mathcal{C} = \{E_{11}, E_{21}, E_{12}, E_{22}\}$

$$E_{11} \xmapsto{f_A} AE_{11} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}$$

$$E_{12} \xmapsto{f_A} AE_{12} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix}$$

$$E_{21} \xmapsto{f_A} \begin{pmatrix} b & 0 \\ d & 0 \end{pmatrix} \qquad E_{22} \xmapsto{f_A} \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}$$

$$[f_A]_C^C = \left(\begin{array}{cc|cc} a & b & 0 & 0 \\ c & d & 0 & 0 \\ \hline 0 & 0 & a & b \\ 0 & 0 & c & d \end{array}\right) = \left(\begin{array}{c|c} A & 0 \\ \hline 0 & A \end{array}\right) = M$$

$$p_M(\lambda) = \det(M - \lambda I_4) = \det\left(\begin{array}{c|c} A - \lambda I_2 & 0 \\ \hline 0 & A - \lambda I_2 \end{array}\right)$$

$$= \left(\det(A - \lambda I)\right)^2 = p_A(\lambda)^2$$

$\Rightarrow$ M e A hanno stessi autovalori

MOLT ALGEBRICHE

λ_i autovalore

$$\Rightarrow \quad p_A(\lambda) = (\lambda - \lambda_i)^{m_a^A(\lambda_i)}$$

$$\Rightarrow \quad p_M(\lambda) = (\lambda - \lambda_i)^{2\, m_a^A(\lambda_i)}$$

$$\Rightarrow \quad m_a^M(\lambda_i) = 2\, m_a^A(\lambda_i)$$

MOLT GEOMETRICHE

$$m_g^M(\lambda_i) = \dim \operatorname{Ker}(M - \lambda_i I) =$$

$$= \dim \operatorname{Ker} \left(\begin{array}{c|c} A-\lambda_i I & 0 \\ \hline 0 & A-\lambda_i I \end{array} \right) =$$

$$\overset{(*)}{=} 2 \dim \operatorname{Ker}(A - \lambda_i I) =$$

$$= 2\, m_g^A(\lambda_i)$$

($\Leftarrow$) A diagonalizzabile, quindi gli autoval di A sono tutti reali e $\forall \lambda \quad m_a^A(\lambda) = m_g^A(\lambda)$

$\Rightarrow \quad m_a^M(\lambda) = 2\, m_a^A(\lambda) = 2\, m_g^A(\lambda) = m_g^M(\lambda)$

$\Rightarrow \quad M$ diag

($\Rightarrow$) Analogo

$\square$

(*) $\dim \operatorname{Ker} \left(\begin{array}{c|c} P & 0 \\ \hline 0 & P \end{array} \right) = 2 \dim \operatorname{Ker} P$
$\forall P \in \operatorname{Mat}(n)$

Dim

$$\dim \operatorname{Ker} \left(\begin{array}{c|c} P & 0 \\ \hline 0 & P \end{array} \right) = 2n - \dim \operatorname{Im} \left(\begin{array}{c|c} P & 0 \\ \hline 0 & P \end{array} \right) =$$

$$= 2n - 2 \dim \operatorname{Im} P = 2 \left(n - \dim \operatorname{Im} P \right) =$$

$$= 2 \dim \operatorname{Ker} P$$

$\square$

(ii) $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow [f_A]_C^C = \left(\begin{array}{cc|cc} 0 & 1 & & \\ 1 & 0 & & 0 \\ \hline & & 0 & 1 \\ & 0 & 1 & 0 \end{array}\right) = M$

$p_A(\lambda) = \lambda^2 - 1 \Rightarrow$ A ha autovalori ± 1

$\Rightarrow$ anche M

$$V^M_{\pm 1} = \mathrm{Ker}(M \mp I) = \mathrm{Ker}\left(\begin{array}{cc|cc} \mp 1 & 1 & & \\ 1 & \mp 1 & & 0 \\ \hline & & \mp 1 & 1 \\ & 0 & 1 & \mp 1 \end{array}\right) = \mathrm{Span}\left(\begin{pmatrix} 1 \\ \pm 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ \pm 1 \end{pmatrix}\right)$$

$\Rightarrow \mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\}$ è una base di autovettori

$\boxed{15}$

$$A = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{pmatrix}$$

Costruire B, se esiste, tale che

(i) B simile ad A

(ii) $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \in \operatorname{Ker} B$

(iii) $\operatorname{Im} B = \{x - y - 2z = 0\}$

$$A \text{ diag} \begin{cases} \lambda = 0 & m_a(0) = 1 \\ \lambda = 1 & m_a(1) = 2 \end{cases} \qquad (\text{esercizio})$$

$$\dim V_0^A = 1 \underset{(1)}{\Longrightarrow} \underset{\substack{\| \\ \dim \operatorname{Ker} B}}{\dim V_0^B} = 1$$

$$\underset{(11)}{\Longrightarrow} \operatorname{Ker} B = \operatorname{Span}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$(111) \Rightarrow \operatorname{Im} B = \operatorname{Span}\left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}\right)$$

Autospazi di B $\begin{cases} V_0 = \text{Span}\begin{pmatrix}1\\0\\1\end{pmatrix} \\ V_1 \text{ di dimensione } 2 \end{cases}$

e $\mathbb{R}^3 = V_0 \oplus V_1$ Scegliamo $\mathcal{B} = \mathcal{B}_{V_0} \cup \mathcal{B}_{V_1} =$

$= \{w_1, w_2, w_3\}$ base di autovettori

$$\left.\begin{aligned} Bw_1 &= 0 \quad \text{perché } V_0 = \operatorname{Ker} B \\ Bw_2 &= w_2 \\ Bw_3 &= w_3 \end{aligned}\right\} \Rightarrow \begin{aligned} &\operatorname{Im} B \\ &\parallel \\ &\text{Span}(w_2, w_3) \\ &\parallel \\ &V_1 \end{aligned}$$

$$V_0 = \mathrm{Span}\begin{pmatrix}1\\0\\1\end{pmatrix}$$

$\Rightarrow$

$$V_1 = \mathrm{Im}\, B = \mathrm{Span}\left(\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}2\\0\\1\end{pmatrix}\right)$$

$B = [f_B]_C^C$, dove $f_B \ \mathbb{R}^3 \to \mathbb{R}^3$ è l'applic lineare tale che $f_B(v) = B\,v$

Quello che sappiamo è che, poiché $\mathcal{B}$ è una base di autovettori per f_B (vedi pag prec)

$$[f_B]_{\mathcal{B}}^{\mathcal{B}} = \begin{pmatrix}0&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$\Rightarrow \quad B = [f_B]_C^C = P \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} P^{-1}, \quad \text{dove } P$$

è la matrice che ha per colonne gli autovettori di B

$$P = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \quad B = \begin{pmatrix} 2 & -1 & -2 \\ 0 & 1 & 0 \\ 1 & -1 & -1 \end{pmatrix}$$