$\boxed{7}$ $U = \{x + y = 0\} \subseteq \mathbb{R}^3$

(1) $p, q \quad \mathbb{R}^3 \to \mathbb{R}^3$ proiez ortogonali su U e $U^\perp$ rispettivamente

$$U = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}^\perp \Rightarrow U^\perp = \operatorname{Span}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \operatorname{Span}\underbrace{\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}}_{v}$$

Teorema dim a lezione (vedi 9 aprile) $\Rightarrow [q]_C^C = v\,v^T =$

$$= \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$p(X) + q(X) = X = id(X) \quad \forall X \in \mathbb{R}^3$$

$$\Rightarrow [p]_c^c = [id]_c^c - [q]_c^c =$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(ii) $f(X) = p(X) - q(X)$

$$[f]^{C}_{C} = [p]^{C}_{C} - [q]^{C}_{C} =$$

$$= \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$[f]^{C}_{C}$ è simmetrica $\Rightarrow$ f è diag per il teorema spettrale

(iii) $A = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

$\langle \, , \rangle_A \quad \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R} \qquad \langle X, Y \rangle_A = X^T A Y$

1. $\langle \, , \rangle_A$ è bilineare $\quad \forall X, X_1, X_2, Y \in \mathbb{R}^3 \ \forall \lambda \in \mathbb{R}$

- $\langle X_1 + X_2 , Y \rangle_A = \langle X_1, Y \rangle_A + \langle X_2, Y \rangle_A$
- $\langle \lambda X, Y \rangle_A = \lambda \langle X, Y \rangle_A$

e analogo sulla seconda componente

- $\langle X_1 + X_2, Y \rangle_A = (X_1 + X_2)^T A Y =$

$$= (X_1^T + X_2^T) A Y = X_1^T A Y + X_2^T A Y =$$

$$= \langle X_1, Y \rangle_A + \langle X_2, Y \rangle_A$$

$$\langle \lambda X, Y \rangle_A = (\lambda X)^T A Y = \lambda X^T A Y =$$

$$= \lambda \langle X, Y \rangle_A$$

$\Rightarrow \langle\, , \rangle_A$ è lineare nella prima componente

$\langle , \rangle_A$ simmetrica $\iff$ A simmetrica

<u>Dim</u> $\langle , \rangle_A$ simm $\iff$ $\forall X, Y \; \langle X, Y\rangle_A = \langle Y, X\rangle_A$

$\iff \forall X, Y \quad X^T A Y = Y^T A X = X^T A^T Y$ ($Y^T A X \in \mathbb{R}$)

$= (Y^T A X)^T = X^T A^T Y$

$\iff \quad A = A^T$ $\square$

PROP $X^T M Y = X^T N Y \quad \forall X, Y \iff M = N$

<u>Dim</u> $(\Leftarrow)$ Ovvia

$(\Rightarrow)$ Scelti $X = e_i$ e $Y = e_j$

si ha $M_{ij} = e_i^T M e_j = e_i^T N e_j = N_{ij}$ $\square$

(iv) $\mathcal{B} = \left(\underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{w_1}, \underbrace{\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}}_{w_2}, \underbrace{\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}}_{w_3} \right)$

M matrice di $\langle , \rangle_A$ risp a $\mathcal{B}$ $\Rightarrow$ $M_{ij} = \langle w_i , w_j \rangle_A$

dalla definizione

$$M_{11} = \langle w_1, w_1 \rangle_A = w_1^T A w_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}^T \begin{pmatrix} -1 & 0 & 1 \\ 0 & -2 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 2$$

Calcolando tutti gli elementi si ha $M = \begin{pmatrix} 2 & 4 & -1 \\ 4 & -3 & 5 \\ -1 & 5 & -6 \end{pmatrix}$

In alternativa

$$w_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = e_1 + e_2 + e_3$$

$$\langle w_1, w_1 \rangle_A = \langle e_1 + e_2 + e_3 , e_1 + e_2 + e_3 \rangle_A =$$

$$= \langle e_1, e_1 \rangle + 2 \langle e_1, e_2 \rangle + 2 \langle e_1, e_3 \rangle$$

$$+ \langle e_2, e_2 \rangle + 2 \langle e_2, e_3 \rangle + \langle e_3, e_3 \rangle =$$

$$= -1 + 2 \; 0 + 2 \; 1 + (-2) + 2 \; 1 + 1 =$$

$$= 2$$

(v) $U^{\perp_A} = \{ X \in \mathbb{R}^3 \quad \langle X, u \rangle_A = 0 \quad \forall u \in U \}$

$U = \text{Span} \left(\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)$

$\Rightarrow U^{\perp_A} = \text{Span} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}^{\perp_A} \cap \text{Span} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}^{\perp_A}$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \text{Span} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}^{\perp_A} \Leftrightarrow$

$0 = \left\langle \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle_A = \langle x e_1 + y e_2 + z e_3, e_3 \rangle_A =$

$= x \langle e_1, e_3 \rangle + y \langle e_2, e_3 \rangle + z \langle e_3, e_3 \rangle = x + y + z$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \operatorname{Span}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}^{\perp_A} \Longleftrightarrow$$

$$\Longleftrightarrow \quad 0 = \left\langle \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\rangle_A =$$

$$= \langle x e_1 + y e_2 + z e_3 , e_1 - e_2 \rangle_A =$$

$$= -x - y(-2) + z - z = -x + 2y$$

$$\Rightarrow \quad U^{\perp_A} = \{ x + y + z = 0 , -x + 2y = 0 \}$$

$\boxed{10}$ $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ matrice di un prodotto scalare $\langle , \rangle_A$

M matrice dell'endomorf (risp base canonica)

M autoagg risp a $\langle , \rangle_A$ $\iff \forall X, Y \ \langle MX, Y \rangle_A = \langle X, MY \rangle_A$

$$\langle MX, Y \rangle_A = X^T M^T A Y \qquad \langle X, MY \rangle_A = X^T A M Y$$

$\iff M^T A = AM$

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$ la cerchiamo non simm e autoaggiunta risp a

$$M^T A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2a+c & a+c \\ 2b+d & b+d \end{pmatrix}$$

$$AM = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2a+c & 2b+d \\ a+c & b+d \end{pmatrix}$$

- M autoagg $\Leftrightarrow M^T A = AM \Leftrightarrow a+c = 2b+d$
- M non simm $\Leftrightarrow b \neq c$

Per esempio $b=-1$ e $c=1$

$\Rightarrow a+1=-2+d$, per esempio $a=0$ e $d=3$ $\Longrightarrow M = \begin{pmatrix} 0 & -1 \\ 1 & 3 \end{pmatrix}$