V sp vettoriale di dim finita con $\langle , \rangle$

$U \subseteq V$ sottospazio vett

$$\Longrightarrow \quad V = U \oplus U^{\perp}$$

<u>CONTROESEMPIO SE dim $V = \infty$</u>

$$V = C([0,1], \mathbb{R}) = \{ f \; [0,1] \to \mathbb{R} \text{ cont} \}$$

dim $V = \infty$ perché $V \supseteq \{ f$ polinomiali $\}$

$$\langle f, g \rangle = \int_0^1 f(x) g(x) \, dx$$

$\langle , \rangle$ è pr scalare

(i) bilineare segue dalla linearità dell'integrale

(ii) simmetrico

$$\langle g, f \rangle = \int_0^1 g(x) f(x)\, dx = \langle f, g \rangle$$

(iii) def positivo

$$\langle f, f \rangle = \int_0^1 (f(x))^2 dx \geq 0 \quad \text{perché } f^2(x) \geq 0 \;\; \forall x \in [0,1]$$

$U = \{ f \in V \quad f(0) = 0 \}$ è sott vettoriale proprio $(U \subsetneq V)$

$U^{\perp} = \{0\}$

<u>Dim</u> Sia $g \in U^{\perp}$ La funzione $x \overset{h}{\mapsto} x\, g(x)$ sta in U

$$\Rightarrow \quad 0 = \langle g, h \rangle = \int_0^1 g(x)\, h(x)\, dx = \int_0^1 x\, g^2(x)\, dx$$

$$\Rightarrow \quad x\, g^2(x) = 0 \quad \forall x \in [0,1]$$

se ci fosse $x_0 \in [0,1]$ con $x_0\, g^2(x_0) > 0$

$\Rightarrow$ è > 0 in un intorno di x_0 $\Rightarrow \int_0^1 x\, g^2(x) > 0$

$\Rightarrow \quad g^2(x) = 0 \quad \forall x \in (0,1]$

$\Rightarrow \quad g(x) = 0 \quad \forall x \in (0,1]$

$\Rightarrow \quad g(x) = 0 \quad \forall x \in [0,1]$

perché g è continua e quindi è continua anche in $x = 0$

$\square$

Tr ortogonali

V sp vett euclideo, $f \; V \to V$ lineare

f ortogonale $\overset{\text{def}}{\iff} \langle f(x), f(y) \rangle = \langle x, y \rangle \quad \forall x, y \in V$

$\overset{\text{teo}}{\iff} \| f(x) \| = \| x \| \quad \forall x \in V$

$\Longrightarrow$ f isometria, ossia preserva le distanze

$A \in M(n)$

<u>A ortogonale</u> $\overset{def}{\iff}$ $A^tA = AA^t = I$

$(A^{-1} = A^t)$

$\underset{teo}{\iff}$ le colonne di A formano una base ortonormale di $\mathbb{R}^n$

<u>Teorema</u> $\mathcal{B}$ base ortonormale di V

f ortogonale $\iff$ $[f]_{\mathcal{B}}^{\mathcal{B}}$ ortogonale

$\boxed{\mathbb{R}^2}$ $A \in M(2)$ ortogonale

$\Rightarrow \exists\, \theta \in [0, 2\pi)$ tale che

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$ rot antioraria di angolo θ

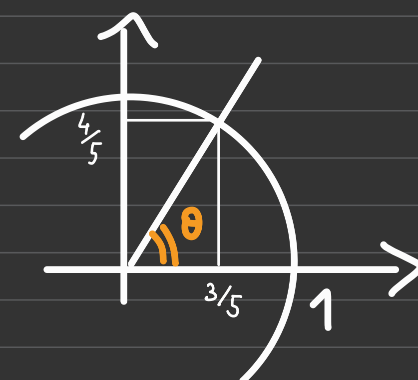
oppure

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$ riflessione risp ad una retta

$\boxed{8}$ $f \ \mathbb{R}^2 \to \mathbb{R}^2$, $A = \begin{pmatrix} 3/5 & a \\ 4/5 & b \end{pmatrix}$

(i) f rot $\Longrightarrow$ $a = -\frac{4}{5}$ e $b = \frac{3}{5}$

$\Longrightarrow A = \begin{pmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{pmatrix}$

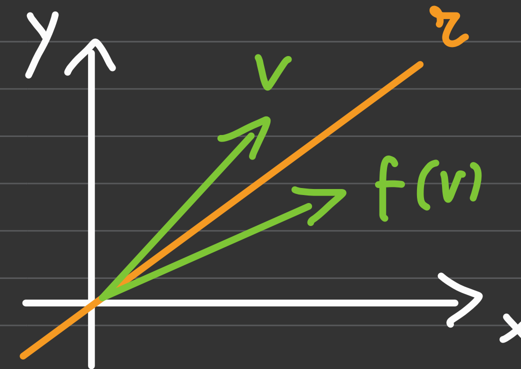


$\cos\theta = \frac{3}{5}$ e $\sin\theta = \frac{4}{5}$

$\Longrightarrow \theta = \arctan \frac{4}{3}$ angolo di rotazione

(ii) f rifl $\Rightarrow$ $a = \frac{4}{5}$ e $b = -\frac{3}{5}$

$$\Rightarrow \quad A = \begin{pmatrix} 3/5 & 4/5 \\ 4/5 & -3/5 \end{pmatrix}$$

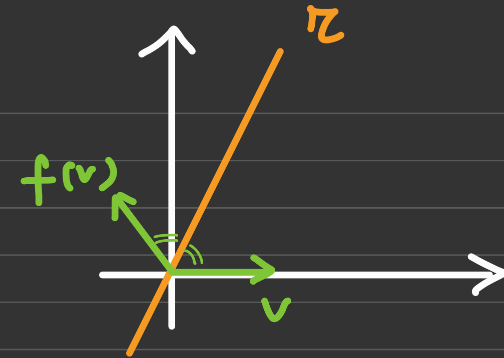


I vettori lungo r sono tutti e soli i $v \in \mathbb{R}^2$ tali che $f(v) = v$, cioè gli autovettori di $\lambda = 1$

$$r = V_1 = \operatorname{Ker}(A - I) = \operatorname{Ker}\begin{pmatrix} -2/5 & 4/5 \\ 4/5 & -8/5 \end{pmatrix} = \operatorname{Span}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

eq cart di r $\quad y = \frac{1}{2}x$

$\boxed{9}^{\text{mod}}$ $\quad r \quad 3x - y = 0$



$f = \text{rifl risp } r$

$[f]_C^C = ?$

$r = \text{Span}\begin{pmatrix}1\\3\end{pmatrix} \qquad r^{\perp} = \text{Span}\begin{pmatrix}3\\-1\end{pmatrix}$

$\mathcal{B} = \left\{ \begin{pmatrix}1\\3\end{pmatrix}, \begin{pmatrix}3\\-1\end{pmatrix} \right\} \Rightarrow [f]_{\mathcal{B}}^{\mathcal{B}} = \left(\begin{array}{c|c}1 & 0\\0 & -1\end{array}\right)$

perché $\quad \begin{pmatrix}1\\3\end{pmatrix} \overset{f}{\longmapsto} \begin{pmatrix}1\\3\end{pmatrix}, \quad \begin{pmatrix}3\\-1\end{pmatrix} \overset{f}{\longmapsto} \begin{pmatrix}-3\\1\end{pmatrix}$

$$[\mathrm{id}]^{B}_{C} = \begin{pmatrix} 1 & 3 \\ 3 & -1 \end{pmatrix} \qquad [\mathrm{id}]^{C}_{B} = \begin{pmatrix} -1 & -3 \\ -3 & 1 \end{pmatrix} \frac{1}{-10}$$

$$\Rightarrow [f]^{C}_{C} = [\mathrm{id}]^{B}_{C} \; [f]^{B}_{B} \; [\mathrm{id}]^{C}_{B} =$$

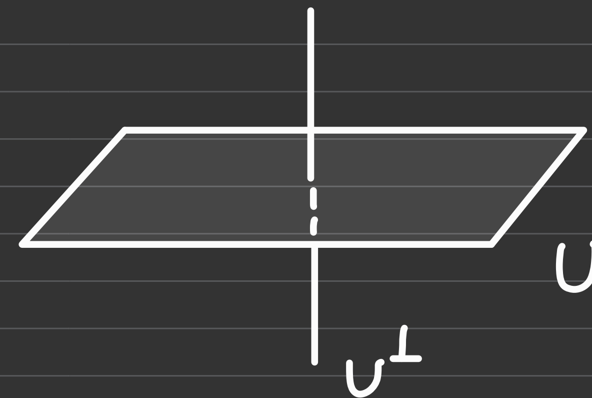
$$= \begin{pmatrix} -4/5 & 3/5 \\ 3/5 & 4/5 \end{pmatrix}$$

$\boxed{12}^{mod}$ U $x+y+z=0$

$f =$ rifl risp a U $[f]_C^C = ?$

$U = \mathrm{Span}\left(\begin{pmatrix}1\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix}\right)$

$U^\perp = \mathrm{Span}\begin{pmatrix}1\\1\\1\end{pmatrix}$



$\mathcal{B} = \mathcal{B}_U \cup \mathcal{B}_{U^\perp}$

$\Rightarrow$ $[f]_{\mathcal{B}}^{\mathcal{B}} = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{array}\right)$ e poi si cambia base

$$[id]^{B}_{C} = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

$$[id]^{C}_{B} = \frac{1}{3}\begin{pmatrix} 2 & -1 & -1 \\ 1 & 1 & -2 \\ 1 & 1 & 1 \end{pmatrix}$$ facendo la matrice inversa

$$\Rightarrow [f]^{C}_{C} = [id]^{B}_{C}\, [f]^{B}_{B}\, [id]^{C}_{B} =$$

$$= \begin{pmatrix} 1/3 & -2/3 & -2/3 \\ -2/3 & 1/3 & -2/3 \\ -2/3 & -2/3 & 1/3 \end{pmatrix}$$