SOLUTIONS OF THE EXERCISES OF WEEK ONE

Exercise 1. For each of the following sentences, state whether they are true or find a counterexample

1. if $n = 1$, for every v, w in E_1 there holds

$$v \cdot w = 0 \Rightarrow v = 0 \text{ or } w = 0$$

2. the same statement as in 1. except that $n \geq 2$.

When $n = 2$, given $v = (v_1, v_2)$ in E_2, we define

$$v^{\perp} := (v_2, -v_1).$$

In the next questions the linear space is E_2.

3. $\|v\| = \|v^{\perp}\|$

4. given v, w such that $w \neq 0$, we have

$$|v \cdot w| + |v \cdot w^{\perp}| = 0 \Rightarrow v = 0$$

5. write the quantity

$$|v \cdot w|^2 + |v \cdot w^{\perp}|^2$$

in terms of $\|v\|$ and $\|w\|$

6.

$$|v \cdot w| = |v \cdot w^{\perp}| \Rightarrow v = w + w^{\perp}.$$

Solution.

1. True. In E_1 the scalar product is the product of two real numbers.

2. False. When $n \geq 2$, the scalar product does not satisfy this property. For instance, take

$$v = (1, 0), \quad w = (0, 1).$$

Clearly, $v \cdot w = 0$, but both vectors are non-zero.

3. True.

$$\|v^{\perp}\|^2 = |(v_2, -v_1) \cdot (v_2, -v_1)| = |v_2^2 + (-v_1)^2| = \|v\|^2.$$

4. True.

$$|v \cdot w| = 0 \Rightarrow v_1 w_1 = -v_2 w_2$$
$$|v \cdot w^{\perp}| = 0 \Rightarrow v_1 w_2 = v_2 w_1.$$

We multiply the first equation by w_1 and the second equation by w_2

$$v_1 w_1 w_2 = -v_2 w_2^2$$
$$v_1 w_2 w_1 = v_2 w_1^2.$$

Now, we subtract the members of the second equation from the first one and obtain

$$0 = -v_2 \|w\|^2.$$

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Now we multiply the first equation by w_1 and the second equation by w_2

$$v_1 w_1^2 = -v_2 w_2 w_1$$
$$v_1 w_2^2 = v_2 w_1 w_2.$$

Taking the sum, we obtain

$$v_1 \|w\|^2 = 0$$

which implies $v_1 = 0$.

5.

$$\begin{aligned}
|v \cdot w|^2 + |v \cdot w^{\perp}|^2 &= |v_1 w_1 + v_2 w_2|^2 + |v_1 w_2 - v_2 w_1|^2 \\
&= (v_1 w_1)^2 + (v_2 w_2)^2 + 2 v_1 w_1 v_2 w_2 \\
&+ (v_1 w_2)^2 + (v_2 w_1)^2 - 2 v_1 w_2 v_2 w_1 \\
&= v_1^2 (w_1^2 + w_2^2) + v_2^2 (w_2^2 + w_1^2) = \|v\|^2 \|w\|^2.
\end{aligned}$$

6. False. Take, for instance $w = 0$ and v any vector different from zero, like $v := (1, 1)$.

□

Exercise 2. Prove the Cauchy-Schwarz inequality

$$|v \cdot w| \leq \|v\| \|w\|$$

for every $v, w \in \mathbb{R}^n$ (start by squaring both terms $|v \cdot w|$ and $\|v\| \|w\|$. It can be easier to look at the case $n = 2$ before the general case).

Solution. Firstly, we consider the case $n = 2$. We have

$$|v \cdot w|^2 = v_1^2 w_1^2 + v_2^2 w_2^2 + 2 v_1 w_1 v_2 w_2.$$

On the other side we have

$$\|v\|^2 \|w\|^2 = v_1^2 w_1^2 + v_2^2 w_2^2 + v_1^2 w_2^2 + v_2^2 w_1^2.$$

So, the inequality holds if and only if

$$2 v_1 w_1 v_2 w_2 \leq v_1^2 w_2^2 + v_2^2 w_1^2.$$

This, is true. It follows from the inequality

$$2ab \leq a^2 + b^2$$

for any real numbers a, b.

Now, let us look at the general case:

$$|v \cdot w|^2 = \sum_{i=1}^{n} v_i^2 w_i^2 + 2 \sum_{j<k} v_j w_j v_k w_k.$$

We apply the inequality with $a = v_j w_j$ and $b = v_k w_k$, and obtain

$$2 \sum_{j<k} v_j w_j v_k w_k \leq \sum_{j<k} (v_j^2 w_k^2 + v_k^2 w_j^2).$$

Then

$$
\begin{aligned}
|v \cdot w|^2 &\leq \sum_{i=1}^{n} v_i^2 w_i^2 + 2 \sum_{j<k} v_j w_j v_k w_k + \sum_{j<k} (v_j^2 w_k^2 + v_k^2 w_j^2) \\
&= \sum_{i=1}^{n} v_i^2 w_i^2 + \sum_{j \neq k} v_j^2 w_k^2 = \sum_{j,k}^{n} v_j^2 w_k^2 = \sum_{j=1}^{n} v_j^2 \cdot \sum_{k=1}^{n} w_k^2 = \|v\|^2 \|w\|^2.
\end{aligned}
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$\square$