EXERCISES OF WEEK FOUR (2014/09/29, 11:00AM)

Exercise 1. Given three vectors $a, b, c \in E_3$, let A be the matrix defined column-wise

$$A := (a|b|c).$$

Show that $\det(A) = a \cdot (b \times c)$.

Solution.

$$\begin{aligned}\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} &= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \\ &= a_1(b_2c_3 - c_2b_3) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - c_1b_2) \\ &= a \cdot b \times c.\end{aligned}$$

□

Exercise 2. Let

$$\ell_1 := \ell(P, v), \quad \ell_2 := \ell(Q, w)$$

be two non-degenerate lines such that $v \times w = 0$. Show that either

$$\ell_1 = \ell_2 \text{ or } \ell_1 \cap \ell_2 = \emptyset.$$

Solution. Since v and w are non-degenerate, there exists c in $\mathbb{R} - \{0\}$ such that $v = cw$. Suppose that

$$\ell_1 \cap \ell_2 \neq \emptyset.$$

Then, there exists R such that R belongs to $\ell_1 \cap \ell_2$. Then

$$\ell_1 = \ell(P, v) = \ell(R, v) = \ell(R, cw) = \ell(R, w) = \ell(Q, w) = \ell_2.$$

□

Exercise 3. Suppose that we have two non-degenerate lines

$$\ell := \ell(P, v), \quad \ell' := \ell(Q, w).$$

in the plane. We can define a distance between ℓ and ℓ'

$$d(\ell, \ell') := \inf\{d(R, R') \mid R \in \ell, R' \in \ell'\}.$$

Try to express the distance in terms of P, Q, v, w.

Solution. If $v \times w \neq 0$, then there exists R in $\ell \cap \ell'$. Hence

$$\text{dist}(\ell, \ell') = 0.$$

Then, suppose that $v \times w = 0$. That is

$$w = cv, \quad c \neq 0.$$

We claim that

$$\text{dist}(P, \ell') = \text{dist}(\ell, \ell').$$

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Clearly, the inequality $\operatorname{dist}(P, \ell') \geq \operatorname{dist}(\ell, \ell')$ holds. We can write

$$\operatorname{dist}(\ell, \ell') = \inf_{R \in \ell} \operatorname{dist}(R, \ell').$$

Given $R \in \ell$, we have

$$\overrightarrow{RQ} = \overrightarrow{RP} + \overrightarrow{PQ}.$$

Since R is in ℓ, there exists t such that

$$\overrightarrow{RP} = tv.$$

Then

$$\begin{aligned} \operatorname{dist}(R, \ell') &= \frac{|\overrightarrow{RQ} \times w|}{\|w\|} = \frac{|\overrightarrow{RQ} \times cv|}{\|cv\|} = \frac{|\overrightarrow{RQ} \times v|}{\|v\|} = \frac{|(\overrightarrow{RP} + \overrightarrow{PQ}) \times v|}{\|v\|} \\ &= \frac{|(\overrightarrow{RP} + \overrightarrow{PQ}) \times v|}{\|v\|} = \frac{|(tv + \overrightarrow{PQ}) \times v|}{\|v\|} = \frac{|\overrightarrow{PQ} \times v|}{\|v\|}. \end{aligned}$$

So,

$$\operatorname{dist}(\ell, \ell') = \inf_{R \in \ell'} \frac{|\overrightarrow{PQ} \times v|}{\|v\|} = \frac{|\overrightarrow{PQ} \times v|}{\|v\|}.$$

□

Exercise 4. Find the area of the polygon with vertices given by the points

$$P(0,0), \quad Q(2,3), \quad R(5,6), \quad T(1,5).$$