SOLUTIONS OF THE EXERCISES OF WEEK FIVE

Exercise 1. Suppose that we have two non-degenerate planes in $\mathbb{R}^3$

$$\pi := \pi(P, u, v), \quad \pi' := \pi(Q, w, z).$$

We can define a distance between π and π' as

$$d(\pi, \pi') := \inf\{d(R, R') \mid R \in \ell, R' \in \ell'\}.$$

Try to express the distance in terms of P, Q, u, v, w, z.

Solution. If $(u \times v) \times (w \times z) \neq 0$, then

$$\pi \cap \pi' \neq \emptyset.$$

On this case, $\operatorname{dist}(\pi, \pi') = 0$. Now, let us suppose that

$$(u \times v) \times (w \times z) \neq 0.$$

Then, there exists $c \in \mathbb{R}$ such that

$$u \times v = cw \times z.$$

On this case, either

$$\pi = \pi'$$

or

$$\pi \cap \pi' \neq \emptyset = 0.$$

On the first case, $\operatorname{dist}(\pi, \pi') = 0$. On the second case, we argue as follows:

$$\operatorname{dist}(\pi, \pi') = \inf_{R \in \pi} \operatorname{dist}(R, \pi').$$

Let R be a point of π. Then there are t, s in $\mathbb{R}$ such that

$$\overrightarrow{PR} = tu + sv.$$

Now,

$$\begin{aligned}
\operatorname{dist}(R, \pi(Q, w, z)) &= \frac{|\overrightarrow{QR} \cdot w \times z|}{\|w \times z\|} = \frac{|(\overrightarrow{QP} + \overrightarrow{PR}) \cdot w \times z|}{\|w \times z\|} \\
&= \frac{|\overrightarrow{QP} \cdot w \times z + (tu + sv) \cdot w \times z|}{\|w \times z\|} \\
&= \frac{|\overrightarrow{QP} \cdot w \times z|}{\|w \times z\|} = \operatorname{dist}(P, \pi).
\end{aligned}$$

Then, $\operatorname{dist}(R, \pi')$ does not depend on R. Hence

$$\operatorname{dist}(\pi, \pi') = \frac{|\overrightarrow{QP} \cdot w \times z|}{\|w \times z\|}.$$

□

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Exercise 2. Suppose that you have a line in $\ell \subseteq \mathbb{R}^2$ given by the equation

$$\ell : ax + by = c.$$

Find a formula for the distance of a point $P(x_0, y_0)$ from ℓ in terms of a, b, c.

Solution. We write the line in parametric form: if $a \neq 0$, then

$$\ell = \ell(Q(c/a, 0), w = (b, -a)).$$

□

We have

$$\begin{aligned}\overrightarrow{PQ} \times (b, -a) &= (x_0 - c/a, y_0) \times (b, -a) \\ &= -a(x_0 - c/a) - by_0 = -(ax_0 + by_0 - c).\end{aligned}$$

Then

$$\text{dist}\big(P, \ell(Q(c/a, 0), (b, -a))\big) = \frac{|\overrightarrow{PQ} \times (b, -a)|}{|(b, -a)|} = \frac{|ax_0 + by_0 - c|}{|\sqrt{a^2 + b^2}|}. \tag{1}$$

If $b \neq 0$, then

$$\ell = \ell(R(0, c/b), (b, -a)).$$

We have

$$\begin{aligned}\overrightarrow{PQ} \times (b, -a) &= (x_0, y_0 - c/b) \times (b, -a) \\ &= -ax_0 - (y_0 - c/b)b = -(ax_0 + by_0 - c).\end{aligned}$$

So the distance formula does not depend whether a or b are different from zero.

Exercise 3. Find the intersection of the two planes given in Cartesian form

$$\pi_1 : 2x + 3y + z = 1 \tag{2}$$

$$\pi_2 : 2x + 3y + 2z = -2. \tag{3}$$

Solution. Since $(2, 3, 1)$ is not parallel to $(2, 3, 2)$, the intersection is a line, which is parallel to

$$(2, 3, 1) \times (2, 3, 2) = (3, -2, 0).$$

Since the determinant of the matrix

$$\begin{pmatrix} 3 & 1 \\ 3 & 2 \end{pmatrix}$$

is non-zero, we can find an intersection point $P(x_0, y_0, z_0)$ with coordinate $x_0 = 0$ and coordinates y_0, z_0 as solution of the system

$$3y_0 + z_0 = 1 \tag{4}$$

$$3y_0 + 2z_0 = -2. \tag{5}$$

We obtain $z_0 = -3$ and $y_0 = 4/3$. Then, the intersection between the two planes is the line

$$\ell(P(0, 4/3, -3), (3, -2, 0)).$$

□

Exercise 4. Given two lines in $\mathbb{R}^2$, $\ell(P,v) \neq \ell(Q,w)$ we know that the intersection is non-empty if and only if $v \times w \neq 0$.

Now, suppose that we have three non-degenerate lines in the plane

$$\ell_1 := \ell(P,v), \quad \ell_2 := \ell(Q,w), \quad \ell_3 := \ell(R,z)$$

such that

$$\ell_j \neq \ell_k \text{ for every } j \neq k$$

find conditions on P, Q, R, v, w, z such that

$$\ell_1 \cap \ell_2 \cap \ell_3 \neq \emptyset.$$

Solution. Since the intersection between the three lines is non-empty, then $\ell_1 \neq \ell_2$. The condition $\ell_1 \neq \ell_2$ implies

$$v \times w \neq 0. \tag{6}$$

So, there is unique intersection point, namely

$$T = P + \frac{\overrightarrow{PQ} \times w}{v \times w} w.$$

In order to have

$$(\ell_1 \cap \ell_2) \cap \ell_3 \neq \emptyset$$

it is necessary that this point T also belongs to ℓ_3. Then $T \in \ell_3$ implies

$$\overrightarrow{TR} \times z = 0.$$

that is

$$\overrightarrow{PR} \times z + \frac{\overrightarrow{PQ} \times w}{v \times w}(w \times z) = 0. \tag{7}$$

The conditions (6) and (7) are sufficient. □