## 1. Applied vectors of the Euclidean space

Given a natural number $n$, we consider the set

$$\mathbb{R}^n := \{(x_1, x_2, \dots, x_n) \mid x_i \in \mathbb{R}\, \forall i\}.$$

We can endow $\mathbb{R}^n$ with a linear structure defined by

$$(v+w)_i := v_i + w_i \tag{1}$$

$$(cv)_i := cv_i \tag{2}$$

for every $v, w \in \mathbb{R}^n$ and $c \in \mathbb{R}$. We also define the *zero vector*

$$(\mathbf{0_n})_i := 0.$$

The use of the bold and the dependence on the dimension prevents any confusion with the zero in $\mathbb{R}$ and the zeroes on spaces of different dimension. However, regardless of the dimension $n$ of the space, we will use the same notation, that is 0, for $\mathbf{0_n}$ and for the zero of the set of real numbers. The distinction will be clear from the context.

**Notation 1.** When we refer to $\mathbb{R}^n$ together with this linear structure, we will use the notation $E_n$.

From the two definitions (1) and (2) it is not difficult to prove the following properties of the sum and the product

$$\begin{aligned}
&\text{(Commutativity)} && v + w = w + v \\
&\text{(Associativity)} && (v+w)+z = v+(w+z) \\
& && (c+d)v = cv + dv \\
& && c(v+w) = cv + cw \\
& && (cd)v = c(dv) \\
&\text{(Additive Identity)} && v + 0 = v.
\end{aligned}$$

Here, we just prove the Commutativity. In order to show that the two vectors are equal, we check that

$$(v+w)_i = (w+v)_i$$

for every $1 \le i \le n$. In fact, we have

$$(v+w)_i = v_i + w_i = w_i + v_i = (w+v)_i.$$

In the first and the third equality we applied (1), while in the second we used the Commutativity Law in the field of real numbers. Likewise, the proof of all the other properties follows by applying (1) and (2) and known properties of the sum and the product between real numbers. Vectors are commonly represented as arrows in the $xy$-plane emanating from the origin (check Figure 1)

When we address the set $\mathbb{R}^n$ together with the linear structure, we will use the notation $E_n$ and call its elements $v, w, z$ *vectors*. Otherwise, we will use the notation $P, Q, R$ and call its elements *points*. A different notation will also be used to specify the coordinates of a point or a vector

$$P(x_1, x_2, \dots, x_n), \quad v := (v_1, v_2, \dots, v_n). \tag{3}$$

Such distinction is useful for modelling, for instance, problems in the Newtonian Mechanics, where forces (represented by vectors) are applied to points (which may correspond to the back of a sledge or the center of a planet). Let us consider the following

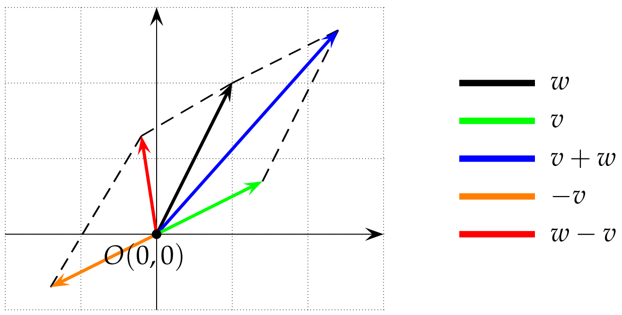



**Figure 1**

product space

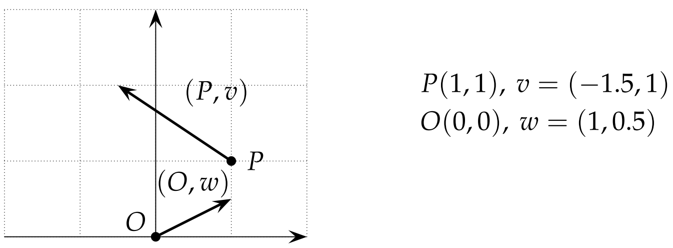
$$E \times \mathbb{R}^n = \{(P, v) \mid P \in \mathbb{R}^n,\, v \in E\}.$$

Having in mind examples from Mechanics, the fact that a force $\mathbf{F}$ is applied at a point $P$, can be represented by the pair $(P, \mathbf{F})$.

**Definition 1** (Applied vectors)**.** We call *applied vector* each of the elements of $E \times \mathbb{R}^n$.

**Definition 2** (Initial point and displacement)**.** Given an applied vector $(v, P)$, we call $P$ *initial point* and $v$ *displacement*.

Applied vectors can be represented as arrows in the $xy$-plane, except that the emanating point is not necessarily the origin.




**Definition 3.** Given two points $P(x_1, x_2, \ldots, x_n)$ and $Q(y_1, y_2, \ldots, y_n)$, we define the *displacement between $P$ and $Q$* the vector

$$\overrightarrow{PQ} := (y_1 - x_1, y_2 - x_2, \ldots, x_n - y_n) \in E_n.$$

Another notation for the displacement is $Q - P$. Given a point $P(x_1, x_2, \ldots, x_n)$ and a vector $v$, we define the point

$$(P + v)_i := x_i + v_i.$$

We call it *endpoint* of the applied vector $(P, v)$.

The following properties can be checked easily from the previous definitions:

**Proposition 1.** *For every $P, Q, R \in \mathbb{R}^n$ and $v, w \in E$, there hold*

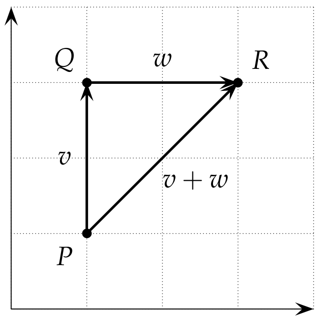
$$P + \overrightarrow{PQ} = Q \tag{4}$$

$$\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR} \tag{5}$$

$$(P + v) + w = P + (v + w) \tag{6}$$

$$(P + v) - P = v. \tag{7}$$

The proof is made by comparing the coordinates of the two points (in the first and third equality) or the two vectors (in the second equality). The third equality have the following geometric interpretation given in the diagram




$Q = (P + v)$
$R = Q + w = P + (v + w)$