3. Cross product

3.1. **Cross product in dimension two.**

Given $v, w \in E_2$ we define the *cross product*

$$v \times w = v_1 w_2 - v_2 w_1 \in E_1.$$

The cross product can be taken as test of pararellism:

Proposition 3.1. *Given $v, w \in E_2$, there holds*

$$v \times w = 0 \Leftrightarrow v \parallel w.$$

Proof.
Proof of $\Leftarrow$. Suppose that $v \parallel w$. Then, according to Definition 2.3, either w is the zero vector or there exists c in $\mathbb{R}$ such that $v = cw$. If $w = 0$, then

$$v \times w = v_1 \cdot 0 - v_2 \cdot 0 = 0.$$

If $v = cw$, then

$$(v \times w) = (cw) \times w = cw_1 w_2 - cw_2 w_1 = 0.$$

Proof of $\Rightarrow$. If $v \times w = 0$, then

$$v_1 w_2 = v_2 w_1.$$

We want to show that $v \parallel w$. If $w = (0,0)$ it is true. Then, we suppose that $w \neq (0,0)$ which means that one between w_1 and w_2 is different from zero. If $w_1 \neq 0$, we have

$$v_2 = \frac{v_1}{w_1} \cdot w_2. \tag{17}$$

We set

$$c := \frac{v_1}{w_1}.$$

Then, from (17)

$$v_2 = cw_2. \tag{18}$$

From

$$v_1 = \frac{v_1}{w_1} \cdot w_1 = cw_1$$

and (18), we can conclude that $v = cw$. If $w_2 \neq 0$ the proof is similar and $c = v_2/w_2$. □

3.2. **Cross product in dimension three.**

Given $v, w, z \in E_3$, we define the *cross product* $v \times w \in E_3$ component-wise as follows

$$(v \times w)_1 := v_2 w_3 - v_3 w_2 \tag{19}$$

$$(v \times w)_2 := v_3 w_1 - v_1 w_3 \tag{20}$$

$$(v \times w)_3 := v_1 w_2 - v_2 w_1. \tag{21}$$

We propose an alternative definition which will be useful for computations.

Notation 3.1 (The Kronecker's delta)**.** Given $1 \leq n$, we define

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Notation 3.2. Given $i, j, k \in \{1,2,3\}$ such that

$$i \neq j, j \neq k, i \neq k$$

we call the symbol (ijk) *permutation*.

Permutations are usually defined as bijections of the set $\{1,2,3\}$ with itself, but for our purposes, permutations will be just symbols.

Given a permutation, for instance (123), we can obtain another permutation, (132), by switching two numbers, here 2 and 3. Such operation is called *transposition*.

Definition 3.1 (Even and odd permutations)**.** A permutation is *even* if we obtain (123) after an even number of transpositions. Otherwise, it is said *odd*.

Example 3.1. From the permutation (321) we obtain (123) with the following steps:

$$(321) \longrightarrow (312) \longrightarrow (132) \longrightarrow (123).$$

In the first step, we transposed 1 with 2. In the second step, 1 with 3. In the third step, 2 with 3. Since three steps were required, (321) is odd; (312) is even (because two steps away from (123)) and (132) is odd; (123) is even. From the transpositions made below

$$(231) \longrightarrow (213) \longrightarrow (123)$$

we obtain that (231) is even and (213) is odd. Since the number of permutations is six, we determined the "evenness" (or the "oddness") of all the permutations.

Before introducing the next symbol, it is useful to notice that the condition

$$i,j,k \in \{1,2,3\}, \quad i \neq j, i \neq k, k \neq i$$

we introduced in Notation 3.2 is equivalent to

$$\{i,j,k\} = \{1,2,3\}. \tag{22}$$

Notation 3.3 (The Levi-Civita symbol)**.** For every $i,j,k \in \{1,2,3\}$ we define

$$\varepsilon_{ijk} = \begin{cases} 0 & \text{if } \{i,j,k\} \neq \{1,2,3\} \\ 1 & \text{if the permutation } (ijk) \text{ is even} \\ -1 & \text{if the permutation } (ijk) \text{ is odd.} \end{cases}$$

Proposition 3.2. *Given $v, w \in E_3$, there holds*

$$(v \times w)_i = \sum_{j=1}^{3} \sum_{k=1}^{3} \varepsilon_{ijk} v_j w_k.$$

for every $1 \leq i \leq 3$.

Proof. We will check the equality only for $i = 1$. The double sum

$$\sum_{j=1}^{3} \sum_{k=1}^{3} \varepsilon_{1jk} v_j w_k \tag{23}$$

has nine different terms. However, most of these are zero. In fact,

$$\varepsilon_{1jk} \neq 0 \Leftrightarrow \{1,j,k\} = \{1,2,3\}.$$

So, we have only the cases

$$j = 2, k = 3 \text{ and } j = 3, k = 2.$$

Then, the term in (23) is equal to

$$\varepsilon_{123} v_2 w_3 + \varepsilon_{132} v_3 w_2 = 1 \cdot v_2 w_3 + (-1) \cdot v_3 w_2.$$

□

We recall some properties of the Levi-Civita symbol:

$$\varepsilon_{ijk} = -\varepsilon_{ikj} = \varepsilon_{kij} \quad \forall i,j,k \tag{24}$$

$$\sum_j \varepsilon_{jik}\varepsilon_{jab} = \delta_{ia}\delta_{kb} - \delta_{ib}\delta_{ka}. \tag{25}$$

Proof. We prove the first equality of (24). We address separately the two cases: when all the indexes are different from each other and when they are not. On the first case, (ijk) and (ikj) are two permutations where the second can be obtained from the first by transposing j with k; this changes the sign of the symbol.
On the second case, both symbols are zero and the equality holds in view of

$$0 = -0.$$

We prove (25). For the sake of simplicity, we call A the left member and B the right member. If $i = k$ (or $a = b$), then ε_{jik} (or ε_{jab}) is equal to zero, while the right member is

$$B = \delta_{ia}\delta_{ib} - \delta_{ib}\delta_{ia} = 0.$$

Now, suppose that $i \neq k$ and $a \neq b$. Suppose that $\{i,k\} \neq \{a,b\}$. Then

$$\{i,k\} \cup \{a,b\} = \{1,2,3\}. \tag{26}$$

We show that

$$\varepsilon_{jik}\varepsilon_{jab} = 0 \ \forall j.$$

Since j is in $\{1,2,3\}$, from (26), either

$$j \in \{i,k\}$$

implying $\varepsilon_{jik} = 0$, or

$$j \in \{a,b\}$$

implying $\varepsilon_{jab} = 0$. Thus, $A = 0$. We show that $B = 0$. Since $\{i,k\} \neq \{a,b\}$, there is an element of the left set which does not belong to the right set. Suppose that such element is i. Then in B, δ_{ia} and $\delta_{ib} = 0$.
Now, suppose that $\{i,k\} = \{a,b\}$. Then, we have two cases: $i = a$ and $k = b$ (then $A = B = 1$) and $i = b$ and $k = a$ (then $A = B = -1$). □

Proposition 3.3. *Given $v, w, z \in E_3$ and $c, d \in \mathbb{R}$, we have*

$$(cv + dw) \times z = c(v \times w) + d(w \times z) \tag{27}$$

$$w \times v = -v \times w. \tag{28}$$

Proof. From Proposition 3.2, we have

$$\begin{aligned}\left[(cv + dw) \times z\right]_i &= \sum_{j,k} \varepsilon_{ijk}(cv + dw)_j z_k \sum_{j,k} \varepsilon_{ijk} c v_j z_k + d w_j z_k \\ &= \sum_{j,k} \varepsilon_{ijk} c v_j z_k + \sum_{j,k} \varepsilon_{ijk} d w_j z_k = c(v \times z)_i + d(v \times w)_i.\end{aligned}$$

As for (28), from (24)

$$(w \times v)_i = \sum_{j,k} \varepsilon_{ijk} w_j v_k = -\sum_{j,k} \varepsilon_{ikj} w_j v_k = -\sum_{j,k} \varepsilon_{ikj} v_k w_j = -(v \times w)_i.$$

□

A consequence of (28) is

$$v \times v = 0. \tag{29}$$

In fact,

$$v \times v = -(v \times v) \Rightarrow 2(v \times v) \Rightarrow v \times v = 0.$$

3.3. **Triple products.** A triple product is any product envolving three vectors and scalar or cross product.

Proposition 3.4 (Triple products). *Given $v, w, z \in E_3$, there holds*

$$(v \times w) \cdot z = (z \times v) \cdot w = (w \times z) \cdot v. \tag{30}$$

$$(v \times w) \times z = (v \cdot z)w - (w \cdot z)v. \tag{31}$$

Proof.

$$\begin{aligned}(v \times w) \cdot z &= \sum_i (v \times w)_i z_i = \sum_i \left(\sum_{j,k} \varepsilon_{ijk} v_j w_k z_i \right) = \sum_k \left(\sum_{i,j} \varepsilon_{ijk} v_j z_i \right) w_k \\ &= \sum_k w_k \sum_{i,j} \varepsilon_{kij} v_j z_i = \sum_k v_k (z \times w) = (z \times w) \cdot v.\end{aligned}$$

The fourth equality follows from (24).

$$\begin{aligned}[(v \times w) \times z]_i &= \sum_{j,k} \varepsilon_{ijk} (v \times w)_j z_k = \sum_{j,k} \varepsilon_{ijk} (v \times w)_j z_k = \sum_{j,k} \varepsilon_{ijk} \sum_{a,b} \varepsilon_{jab} v_a w_b \\ &= \sum_{j,k} \sum_{a,b} \varepsilon_{ijk} \varepsilon_{jab} v_a w_b z_k = -\sum_{k,a,b} v_a w_b z_k \sum_j \varepsilon_{jik} \varepsilon_{jab}.\end{aligned}$$

In the last equality we applied (24). Now, we apply (25). Then the computation carries on as follows.

$$-\sum_{k,a,b} (\delta_{ia}\delta_{kb} - \delta_{ib}\delta_{ka}) v_a w_b z_k = \sum_{k,a,b} \delta_{ib}\delta_{ka} v_a w_b z_k - \sum_{k,a,b} \delta_{ia}\delta_{kb} v_a w_b z_k = A - B.$$

We address the two terms separately:

$$\begin{aligned}A &= \sum_k z_k \sum_{a,b} \delta_{ib}\delta_{ka} v_a w_b = \sum_k z_k \sum_a \delta_{ka} v_a \sum_b \delta_{ib} w_b = \sum_k z_k v_k w_i = (v \cdot z) w_i \\ B &= \sum_k z_k \sum_{a,b} \delta_{ia}\delta_{kb} v_a w_b = \sum_k z_k \sum_a \delta_{ia} v_a \sum_b \delta_{kb} w_b = \sum_k z_k v_i w_k = (w \cdot z) v_i.\end{aligned}$$

Then

$$[(v \times w) \times z]_i = (v \cdot z) w_i - (w \cdot z) v_i.$$

□

From (30) it follows that $v \times w$ is orthogonal to v and w:

$$(v \times w) \cdot v = (v \times v) \cdot w = 0 \cdot w = 0$$

from (29). Since the equality above holds for every v and w, we can switch the roles of v and w with each other and obtain $(v \times w) \cdot w = 0$. We can use (30) and (31) to evaluate the norm of the cross product: given two vectors $v, w \in E_3$, we have

$$\begin{aligned}\|v \times w\|^2 &= (v \times w) \cdot (v \times w) = ((v \times w) \times v) \cdot w \\ &= ((v \cdot v)w - (v \cdot w)v) \cdot w = \|v\|^2 \|w\|^2 - |(v \cdot w)|^2.\end{aligned} \tag{32}$$

Proposition 3.5. *Two vectors $v, w \in E_3$ are parallel if and only if $v \times w = 0$.*

Proof. If v and w are parallel, then either $w = 0$ (in which case the conclusion follows rightaway) or there exists a real number c such that

$$v = cw.$$

Then

$$v \times w = c(w \times w) = 0$$

by (29). Now, suppose that $v \times w = 0$. Then, the norm of $v \times w$ is zero. From (32)

$$\|v\|^2\|w\|^2 = |(v \cdot w)|^2.$$

Then v is parallel to w by Proposition 2.1. □

3.4. **Geometric interpretations of the cross product.** The cross product can be used to evaluate area of polygons. Given two points P, Q in $\mathbb{R}^n$, we will use the notation

$$\overline{PQ} := \|\overrightarrow{PQ}\|.$$

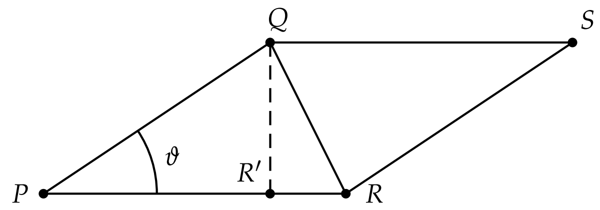

Figure 3. Area of the triangle and the parallelogram

From (32), we have

$$\|v \times w\|^2 = \|v\|^2\|w\|^2 \left(1 - \frac{|(v \cdot w)|^2}{\|v\|^2\|w\|^2}\right) = \|v\|^2\|w\|^2(1 - \cos^2 \vartheta)$$
$$= \|v\|^2\|w\|^2 \sin^2 \vartheta.$$

Then

(33) $$\|v \times w\| = \|v\|\|w\|| \sin \vartheta|$$

From Figure 3, the area of the parallelogram $PQSR$ is equal to

$$\overline{PR} \cdot \overline{R'Q} = \overline{PR} \cdot \overline{PQ}| \sin \vartheta|.$$

By (33),

$$\overline{PR} \cdot \overline{PQ}| \sin \vartheta| = \overline{PR} \cdot \overline{PQ} \cdot \frac{\|\overrightarrow{PR} \times \overrightarrow{PQ}\|}{\overline{PR} \cdot \overline{PQ}} = \overline{PR} \cdot \overline{PQ}.$$

So,

(34) $$Area(PQSR) = \|\overrightarrow{PR} \cdot \overrightarrow{PQ}\|.$$

The area of the triangle PQR is half the area of the parallelogram $PQRS$. Then

$$Area(PQR) = \frac{Area(PQRS)}{2} = \frac{1}{2} \cdot \|\overrightarrow{PR} \times \overrightarrow{PQ}\|.$$

Looking at the Figure 4 we can show that the volume of a parallelepiped is related to the triple product (30).

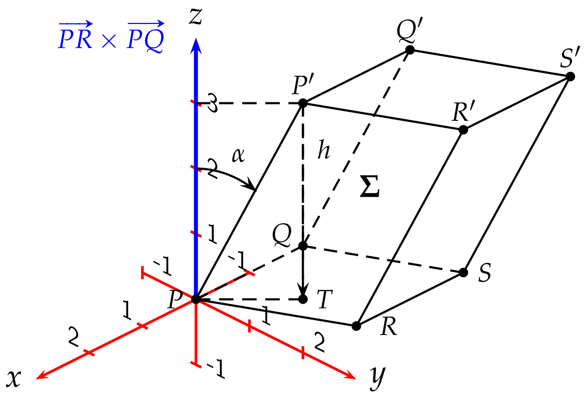

Figure 4. Area of a parallelepiped

$$Vol(\mathbf{\Sigma}) = Area(PQRS) \cdot h.$$

From (34),

$$Area(PQRS) = \|\overrightarrow{PQ} \times \overrightarrow{PR}\|$$

while

$$h = \overline{PP'}|\cos\alpha|.$$

Since α is the angle between $\overrightarrow{PP'}$ and $\overrightarrow{PR} \times \overrightarrow{PQ}$, we have

$$|\cos\alpha| = \frac{(\overrightarrow{PQ} \times \overrightarrow{PR}) \cdot \overrightarrow{PP'}}{\|\overrightarrow{PQ} \cdot \overrightarrow{PR}\| \cdot \overline{PP'}}$$

Then

$$Vol(\Sigma) = \|\overrightarrow{PQ} \times \overrightarrow{PR}\| \cdot \overline{PP'} \cdot \frac{(\overrightarrow{PQ} \times \overrightarrow{PR}) \cdot \overrightarrow{PP'}}{\|\overrightarrow{PQ} \times \overrightarrow{PR}\| \cdot \overline{PP'}} = (\overrightarrow{PQ} \times \overrightarrow{PR}) \cdot \overrightarrow{PP'}.$$