5. Intersection of planes and lines

Definition 5.1 (Parametric form)**.** Given a point $P \in \mathbb{R}^3$ and vectors $v, w \in E^3$, a *plane* is the set

$$\pi(P, v, w) := \{P + tv + sw \mid t, s \in \mathbb{R}\}.$$

Definition 5.2. A plane is called *non-degenerate* if and only if

$$v \times w \neq 0.$$

Proposition 5.1. *Let $\ell_1 := \ell(P, v)$ and $\ell_2 := \ell(Q, w)$ be two non-degenerate lines such that $\ell_1 \neq \ell_2$. Then $\ell_1 \cap \ell_2 \neq \emptyset$ if and only if*

$$\overrightarrow{PQ} \cdot (v \times w) = 0, \quad v \times w \neq 0. \tag{42}$$

If the intersection is non-empty, it consists of a single point

$$R = Q + \frac{(\overrightarrow{PQ} \times v) \cdot (v \times w)}{\|v \times w\|^2} w = P + \frac{(\overrightarrow{PQ} \times w) \cdot (v \times w)}{\|v \times w\|^2} v \tag{43}$$

Proof. Suppose that $\ell_1 \cap \ell_2 \neq \emptyset$. Then, there exists R such that

$$R \in \ell_1 \cap \ell_2.$$

Then, there are real numbers t, s such that

$$R = P + tv, \quad R = Q + sw.$$

Then

$$\overrightarrow{PQ} = tv - sw \Rightarrow \overrightarrow{PQ} \cdot (v \times w) = 0.$$

We can show by contradiction that $v \times w \neq 0$. If $v \times w = 0$, then there exists $c \neq 0$ such that

$$w = cv.$$

Then

$$\begin{aligned} \ell_1 &= \ell(Q, w) = \ell(P + \overrightarrow{PQ}, w) \\ &= \ell(P + tv - sw, w) = \ell(P + (t - cs)v, cv) \\ &= \ell(P, v) = \ell_2 \end{aligned}$$

which contradicts the assumption that $\ell_1 \neq \ell_2$.

Conversely, suppose that (46) holds. Then, $\overrightarrow{PQ}$ is a vector of the linear space generated by v and w. Then there are α and β such that

$$\overrightarrow{PQ} = \alpha v + \beta w \tag{44}$$

that is

$$Q - P = \alpha v + \beta w$$

whence

$$Q - \beta w = P + \alpha v. \tag{45}$$

Thus, if we define $R := P + \alpha v$, we have $R \in \ell_1 \cap \ell_2$.

Now, we wish to find an explicit formula for the intersection point. That is, in (44) we wish to find α and β. We take the cross product with w and obtain

$$\overrightarrow{PQ} \times w = \alpha v \times w$$

then

$$(\overrightarrow{PQ} \times w) \cdot (v \times w) = \alpha \|v \times w\|^2.$$

Since $v \times w \neq 0$, we have

$$\alpha = \frac{(\overrightarrow{PQ} \times w) \cdot (v \times w)}{\|v \times w\|^2}.$$

Then the intersection point is

$$R = P + \frac{(\overrightarrow{PQ} \times w) \cdot (v \times w)}{\|v \times w\|^2} v$$

By taking the cross product with v, we obtain

$$\overrightarrow{PQ} \times v = \beta w \times v$$

whence

$$\beta = -\frac{(\overrightarrow{PQ} \times v) \cdot (v \times w)}{\|v \times w\|^2} w.$$

From (45) the equality (43) follows. □

In the next proposition we find an explicit formula for the intersection of a line $\ell(P, v)$ with a plane $\pi(Q, w, z)$. If $\ell \subseteq \pi$, then the intersection $\ell \cap \pi$ is equal to ℓ.

Proposition 5.2 (Intersection between a line and a plane)**.** *Suppose that $\ell \not\subseteq \pi$. Then $\ell \cap \pi \neq \emptyset$ if and only if*

$$w \times z \cdot v \neq 0.$$

On this case, the intersection is a single point R and

$$R = P + \left(\frac{\overrightarrow{PQ} \cdot w \times z}{z \cdot w \times z} \right) v.$$

Proof. If the intersection is non-empty, then $v \cdot (w \times z) \neq 0$.
Let R be a point of the intersection. We show that

$$v \times w \cdot z \neq 0.$$

On the contrary, there are α and β such that

(46) $$z = \alpha v + \beta w.$$

Then

$$\ell(P, v) = \ell(R, v) = \ell(R, \alpha w + \beta z) \subseteq \pi(R, w, z) = \pi(Q, w, z).$$

And we obtain a contradiction. Now, we prove that if $v \cdot (w \times z) \neq 0$, then the intersection is non-empty. We have to prove that there are t, s, r such that

$$R = Q + sw + rz, \quad R = P + tz$$

or

$$\overrightarrow{PQ} = tv - sw - rz.$$

Such real numbers exist, because $\{v, w, z\}$ is a basis of E_3. Now, we wish to evaluate t to find an explicit formula for the intersection point. We set

$$\alpha := t, \quad \beta := -s, \quad \gamma := -r$$

and

$$b := \overrightarrow{PQ}.$$

Then, we need to find real numbers α, β, γ such that

$$b = \alpha v + \beta w + \gamma z.$$

To this purpose, we use the Cramer's Rule:

$$\alpha = \frac{b \cdot w \times z}{v \cdot w \times z}, \quad \beta = \frac{v \cdot b \times z}{v \cdot w \times z}, \quad \gamma = \frac{v \cdot w \times b}{v \cdot w \times z}.$$

Then, the intersection point is

$$R = P + \left(\frac{\overrightarrow{PQ} \cdot w \times z}{v \cdot w \times z} \right) v = Q - \left(\frac{v \cdot \overrightarrow{PQ} \times z}{v \cdot w \times z} \right) w - \left(\frac{v \cdot w \times \overrightarrow{PQ}}{v \cdot w \times z} \right) z.$$

□

Given $Q \in \mathbb{R}^3$ and a plane $\pi(P, w, z)$. We define the distance between Q and π as

$$d(Q, \pi) := \inf\{d(Q, R) \mid R \in \pi\}.$$

Proposition 5.3 (Distance between a point and a plane)**.** *Given* $Q \in \mathbb{R}^3$ *and a non-degenerate plane* $\pi(P, w, z)$*. Then*

$$d(Q, \pi) = \frac{|\overrightarrow{PQ} \cdot w \times z|}{\|w \times z\|}$$

Proof. We define

$$\ell' := \ell(Q, v \times w).$$

By Proposition 5.2, the intersection between ℓ' and π consists of a single point

$$Q' = Q - \left(\frac{v \times w \cdot \overrightarrow{PQ}}{\|v \times w\|^2} \right) v \times w.$$

Then

$$d(Q, \pi) \leq \|\overrightarrow{QQ'}\| = \frac{|v \times w \cdot \overrightarrow{PQ}|}{\|v \times w\|} \tag{47}$$

Now, let R be another point of π. Then

$$\overrightarrow{RQ} \cdot \overrightarrow{QQ'} = 0.$$

Hence

$$d(Q, R)^2 = d(Q, Q')^2 + d(R, Q')^2$$

and

$$d(Q, R) \geq d(Q, Q') = \frac{|v \times w \cdot \overrightarrow{PQ}|}{\|v \times w\|}.$$

Since the inequality above holds for every $R \in \pi$, we obtain

$$d(Q, \pi) \geq \frac{|v \times w \cdot \overrightarrow{PQ}|}{\|v \times w\|}.$$

Together with (47), we obtain

$$d(Q, \pi) = \frac{|v \times w \cdot \overrightarrow{PQ}|}{\|v \times w\|}.$$

□