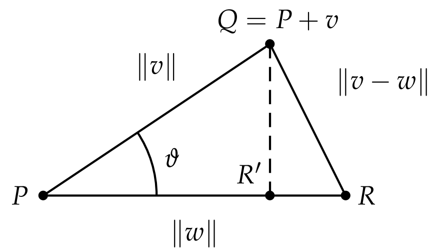

Figure 2. Scalar product in terms of $\cos\vartheta$

2. Scalar product in Euclidean spaces

In this section, we introduce the definition of *segment*. Given two points $P, Q \in \mathbb{R}^n$, the segment between P and Q is a subset of $\mathbb{R}^n$ defined as

$$\{P + t\overrightarrow{PQ} \mid 0 \le t \le 1\} \subseteq \mathbb{R}^n.$$

Definition 2.1. Given two vectors $v, w \in E$, we define the real number

$$v \cdot w := \sum_{i=1}^{n} v_i w_i.$$

It is called *scalar product* or *dot product*.

The scalar product satisfies the following equalities for every $v, w, z \in E_n$ and $c, d \in \mathbb{R}$

$$(cv + dw) \cdot z = cv \cdot z + cv \cdot z \tag{8}$$

$$v \cdot w = w \cdot v \tag{9}$$

$$v \cdot v \ge 0 \text{ and } v \cdot v = 0 \Leftrightarrow v = 0. \tag{10}$$

Definition 2.2 (Norm and unit vectors)**.** Given $v \in E$ we define the *norm* of v as $\|v\| := \sqrt{v \cdot v}$. A vector $w \in E$ is a *unit vector* if $\|w\| = 1$.

We can always write a vector $v \neq 0$ as product of a real number and a unit vector

$$v = \frac{v}{\|v\|} \cdot \|v\|. \tag{11}$$

The norm of a vector (also called *magnitude*) can be represented as the length of the segment between P and $P + v$; the scalar product $v \cdot w$ has a geometric interpretation in terms of the cosinus of the angle between v and w.

In Figure 2 we wrote the length of each side of the triangle PQR. By the Cosinus Theorem, there holds

$$\|v - w\|^2 = \|v\|^2 + \|w\|^2 - 2\|v\|\,\|w\| \cos\vartheta$$

whence

$$\|v\|^2 + \|w\|^2 - 2v \cdot w = \|v\|^2 + \|w\|^2 - 2\|v\|\,\|w\| \cos\vartheta$$
$$\Rightarrow v \cdot w = \|v\|\,\|w\| \cos\vartheta.$$

If $\|v\|\,\|w\| > 0$, then

$$\cos\vartheta = \frac{v \cdot w}{\|v\|\,\|w\|}.$$

Definition 2.3 (Parallel and orthogonal vectors)**.**

(i) Two vectors $v, w \in E$ are *parallel* to each other if either $w = 0$ or there exists $c \in \mathbb{R}$ such that $v = cw$. We use the notation $v \parallel w$

(ii) v is *orthogonal* to w if and only if $v \cdot w = 0$. We use the notation $v \perp w$.

Proposition 2.1 (The Cauchy-Schwarz inequality)**.** *Given $v, w \in \mathbb{R}^n$ there holds*

(a) $|v \cdot w| \leq \|v\| \|w\|$

(b) if the equality holds and $w \neq 0$, then there exists c in $\mathbb{R}$ such that $v = cw$.

Before giving the proof of this proposition, we notice that the geometric interpretation of the cosinus provides us with a proof: (a) follows from the fact that $|\cos\vartheta| \leq 1$; if the equality holds, we have $\cos\vartheta = \pm 1$ which means that ϑ is a multiple of π and (b) follows.

Now, we give a proof based only on the definition of the scalar product without any appeal to the geometric intuition.

Proof of the Cauchy-Schwarz inequality. If $w = 0$, then the inequality turns into $0 \leq 0$, which is true. Suppose that $w \neq 0$. Then, we define

$$a = v - \left(\frac{v \cdot w}{\|w\|^2} \right) w.$$

In the Figure 2, a corresponds to the vector $\overrightarrow{R'Q}$.

$$A := \|a\|^2 \tag{12}$$

is non-negative from property (10). We have

$$0 \leq A = \|v\|^2 + \frac{(v \cdot w)^2}{\|w\|^4} \|w\|^2 - 2\frac{(v \cdot w)^2}{\|w\|^2} = \|v\|^2 - \frac{(v \cdot w)^2}{\|w\|^2}. \tag{13}$$

Then

$$\|v\|^2 - \frac{(v \cdot w)^2}{\|w\|^2} \geq 0 \tag{14}$$

which implies

$$\|v\|^2 \|w\|^2 \geq |v \cdot w|^2 \tag{15}$$

whence

$$\|v\| \|w\| \geq |v \cdot w|. \tag{16}$$

If the equality holds in (16), then the term in (14) is equal to zero. Then, from (12), $A = 0$. Again, by property (10),

$$v = \frac{v \cdot w}{\|w\|^2} w$$

so $v := cw$ with the choice

$$c = \frac{v \cdot w}{\|w\|^2}$$

implying, again $v \parallel w$. □

Proposition 2.2 (The triangular inequality)**.** *Given $v, w \in E_n$ there holds*

$$\|v + w\| \leq \|v\| + \|w\|.$$

Proof. We take the square of $\|v+w\|$ and obtain

$$\begin{aligned}\|v+w\|^2 &= (v+w)\cdot(v+w) = \|v\|^2 + \|w\|^2 + 2v\cdot w \\ &\leq \|v\|^2 + \|w\|^2 + 2\|v\|\|w\|.\end{aligned}$$

□

Such inequality takes its name from the following geometric property: given a triangle PQR, each edge is smaller than the sum of the two other edges:

$$\|\overrightarrow{PQ}\| = \|\overrightarrow{PR} + \overrightarrow{RQ}\| \leq \|\overrightarrow{PR}\| + \|\overrightarrow{RQ}\|.$$