SOLUTIONS OF THE EXERCISES OF WEEK TEN

Exercise 1 (page 127, ex. 1).

$$y'' + 3y' + 2y = 6$$

Solution.

$$p(X) = X^2 + 3X + 2 \quad r_1 = -1, r_2 = -2$$

then

$$y_0 = c_1 e^{-x} + c_2 e^{-2x}.$$

We choose a particular solution among the constants $y_p(x) = c$

$$y_p = 3$$

Then

$$y = 3 + c_1 e^{-x} + c_2 e^{-2x}.$$

□

Exercise 2 (page 127, ex. 2).

$$4y'' + 9y = 15$$

Solution.

$$p(X) = 4X^2 + 9$$

then

$$y_0 = c_1 \cos \frac{3x}{2} + c_2 \sin \frac{3x}{2}.$$

The non-homogeneous term $g = 15$ is not a solution of the homogeneous equation. Then, we choose $y_p = c$

$$c = \frac{5}{3}$$

and

$$y = \frac{5}{3} + c_1 \cos \frac{3x}{2} + c_2 \sin \frac{3x}{2}.$$

□

Exercise 3 (page 127, ex. 3).

$$y'' - 10y' + 25y = 30x + 3$$

Solution. The solution of the homogeneous equation is

$$y_0 = c_1 e^{5x} + c_2 x e^{5x}.$$

We choose $y_p = Ax + B$. Then

$$-10A + 25Ax + 25B = 30x + 3$$

then

$$y_p = \frac{3}{5}(2x + 1)$$

and

$$y = \frac{3}{5}(2x + 1) + c_1 e^{5x} + c_2 x e^{5x}.$$

□

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Exercise 4 (page 127, ex. 4).

$$y'' + y' - 6y = 2x$$

Solution.

$$y_0 = c_1 e^{2x} + c_2 e^{-3x}$$
$$y_p = Ax + B$$
$$Ly_p = A - (6Ax + 6B) = 2x \Rightarrow A = -\frac{1}{3}, B = -\frac{1}{18}$$

then

$$y = \frac{1}{18}(6x+1) + c_1 e^{2x} + c_2 e^{-3x}.$$

□

Exercise 5 (page 127, ex. 6).

$$y'' - 8y' + 20y = 100x^2 - 26xe^x$$

Solution.

$$y_0 = c_1 e^{4x}\cos(2x) + c_2 e^{4x}\sin(2x)$$

the non-homogeneous term is not a solution of the homogeneous equation. Then, we can choose

$$y_p^1 = Ax^2 + Bx + C, \quad y_p^2 = (Dx+E)e^x.$$

We obtain

$$y = 5x^2 + 4x + \frac{11}{10} - e^x\left(2x + \frac{12}{13}\right)$$

□

Exercise 6 (page 127, ex. 8).

$$4y'' - 4y' - 3y = \cos(x)$$

Solution.

$$y_0 = c_1 e^{-x/2} + c_2 e^{3x/2}$$
$$y_p = -\frac{1}{65}\left(7\cos(x) + 4\sin(x)\right)$$

□

Exercise 7 (page 127, ex. 10).

$$y'' + 2y' = 2x + 5 + e^{-2x}$$

Solution.

$$y_0 = c_1 + c_2 e^{-2x}.$$

We will find two particular solutions $y_{p,1}$ and $y_{p,2}$ such that

$$Ly_{p,1} = 2x + 5, \quad Ly_{p,2} = e^{-2x}.$$

For $y_{p,1}$ we try a polynomial of degree two We have

$$y_{p,1} = Ax^2 + Bx$$

Then

$$L(Ax^2 + Bx) = (Ax^2 + Bx)'' + 2(Ax^2 + Bx)' = 2A + 4Ax + 2B = 2x + 5.$$

Then

$$A = \frac{1}{2}, \quad B = 2.$$

Then

$$y_{p,1} = \frac{x^2}{2} + 2x.$$

As for $y_{p,2}$, we cannot choose Ce^{-2x}, because e^{-2x} is a solution to the homogeneous equation. Then we choose

$$y_{p,2}(x) = Cxe^{-2x}.$$

We have

$$\begin{aligned} L(xe^{-2x}) &= (D+2)D(xe^{-2x}) = (D+2)(e^{-2x}(1-2x)) \\ &= D(e^{-2x}(1-2x)) + 2e^{-2x}(1-2x) \\ &= e^{-2x}(-2-2(1-2x)) + 2e^{-2x}(1-2x) = e^{-2x}(-2-2+4x) + e^{-2x}(2-4x) \\ &= -2e^{-2x}. \end{aligned}$$

Then

$$C = -\frac{1}{2}.$$

$$y_p = \frac{x^2}{2} + 2x - \frac{1}{2}e^{-2x}.$$

□

Exercise 8 (page 127, ex. 18)**.**

$$y'' - 2y' + 2y = e^{2x}(\cos x - 3\sin x)$$

Solution.

$$y_0 = c_1 e^x \cos x + c_2 e^x \sin x.$$

Since g is not a solution to the homogeneous equation we can find a particular solutions as

$$y_p = Ae^{2x}\cos x + Be^{2x}\sin x.$$

We have

$$\begin{aligned} y_p &= e^{2x}\big[A\cos x + B\sin x\big] \\ y_p' &= e^{2x}\big[\cos x(2A+B) + \sin x(2B-A)\big] \\ y_p'' &= e^{2x}\big[\cos x(3A+4B) + \sin x(3B-4A)\big]. \end{aligned}$$

Then

$$\begin{aligned} y_p'' - 2y_p' + 2y_p &= e^{2x}\cos x\Big[(3A+4B) - 2(2A+B) + 2A\Big] \\ &\quad + e^{2x}\sin x\Big[(3B-4A) - 2(2B-A) + 2B\Big] \\ &= e^{2x}\cos x(2B+A) + e^{2x}\sin x(B-2A). \end{aligned}$$

Then we have to solve the system

$$\begin{cases} 2B + A &= 1 \\ B - 2A &= -3 \end{cases}$$

whence

$$A = \frac{7}{5}, \quad B = -\frac{1}{5}.$$

So, the general solution is

$$y = c_1 e^x \cos x + c_2 e^x \sin x + \frac{1}{5}\left(7e^{2x}\cos x - e^{2x}\sin x\right).$$

□

Exercise 9 (page 127, ex. 24).

$$y^{(3)} - y'' - 4y' + 4y = 5 - e^x + e^{2x}$$

Solution.

$$y_0 = c_1 e^x + c_2 e^{2x} + c_3 e^{-2x}$$
$$y_p = \frac{5}{4} + \frac{xe^x}{3} + \frac{xe^{2x}}{4}.$$

□

Exercise 10 (page 127, ex. 26).

$$y^{(4)} - y'' = 4x + 2xe^{-x}.$$

Solution.

$$y_0 = c_1 + c_2 x + c_3 e^x + c_4 e^{-x}$$

Since x and xe^{-x} are solutions to the homogeneous equation, we try to find a solution to the non-homogeneous equation as

$$y_p = y_{p,1} + y_{p,2}.$$

The first choice for $y_{p,1}$ would be a second degree polynomial where

$$y_{p,1} = Ax^2 + Bx + C.$$

However,

$$Ly_{p,1} = L(Ax^2) + L(Bx + c) = -2A \neq 4x$$

for every choice of A. Then we choose

$$y_{p,1} = Ax^3 + Bx^2.$$

We obtain

$$Ly_{p,1} = L(Ax^3 + Bx^2) = -D^2(Ax^3) - D^2(Bx^2) = -6Ax - 2B.$$

Thus,

$$B = 0, \quad A = -\frac{2}{3}$$

and

$$y_{p,1} = -\frac{2}{3}x^3.$$

For $y_{p,2}$, we try

$$Exe^{-x}.$$

However,

$$L(Exe^{-x}) = -2Exe^{-x}$$

Then, as second attempt, we choose

$$y_{p,2} = (Fx^2 + Ex)e^{-x}.$$

We have

$$L(Fx^2 + Exe^{-x}) = L(Fx^2 e^{-x}) + L(Ex)e^{-x} = L(Fx^2 e^{-x}) - 2Ee^{-x}.$$

In order to evaluate the first term in the equality above, we apply two times the Leibniz rule to the second order

(1) $$(fg)'' = f''g + 2f'g' + fg''.$$

Then

$$D^2(x^2 e^{-x}) = e^{-x}(x^2 - 4x + 2)$$

and

$$D^4(x^2e^{-x}) = D^2\Big[e^{-x}(x^2-4x+2)\Big] = e^{-x}\big[x^2-4x+2-2(2x-4)+2\big]$$
$$= e^{-x}(x^2-8x+12)$$

Then

$$L(Ex^2e^{-x}) = Ee^{-x}(x^2-8x+12) - Ee^{-x}(x^2-4x+2) = Ee^{-x}(-4x+10).$$

Then

$$L(Ex^2e^{-x}+Fxe^{-x}) = Ee^{-x}(-4x+10) - 2Fe^{-x} = e^{-x}(10E-2F) - 4Exe^{-x}.$$

This should be equal to

$$2xe^{-x}.$$

Then, E and F must be a solution of the system

$$\begin{cases} 10F - 2E &= 0 \\ -4F &= 2 \end{cases}$$

Then

$$F = -\frac{1}{2}, \quad E = 10.$$

Then

$$y_{p,2} = -\frac{1}{2}(x^2+5x)e^{-x}.$$

So, the solution to the differential equation is

$$y = -\frac{1}{2}(x^2+5x)e^{-x} - \frac{2}{3}x^3 + c_1 + c_2x + c_3e^x + c_4e^{-x}.$$

□

Exercise 11. Find all the solutions of the differential equation

$$(D^2+1)^2y = 0$$

Solution. We use the substitution

$$(D^2+1)y = z.$$

Then

$$z(x) = c_1\cos x + c_2\sin x.$$

Now, we need to find a solution to

$$(D^2+1)y = c_1\cos x + c_2\sin x.$$

The solution to the homogeneous equation is

$$y_0 = c_3\cos x + c_4\sin x.$$

Since $\sin x$ and $\cos x$ are solution to the homogeneous equation, we will search a solution as

$$Ax\cos x + Bx\sin x.$$

We have

$$(D^2+1)(Ax\cos x + Bx\sin x) = D^2(Ax\cos x + Bx\sin x) + (Ax\cos x + Bx\sin x)$$

Again, we apply the Leibniz formula in (1). Then

$$=D^2(Ax\cos x) + D^2(Bx\sin x) + (Ax\cos x + Bx\sin x)$$
$$=A(-2\sin x - x\cos x) + B(2\cos x - x\sin x) + (Ax\cos x + Bx\sin x) = -2A\sin x + 2B\cos x.$$

Then

$$A = -\frac{c_2}{2}, \quad B = \frac{c_1}{2}.$$

Then the solutions are

$$y = -\frac{c_2}{2} x \cos x + \frac{c_1}{2} x \sin x + c_3 \cos x + c_4 \sin x$$

for every c_1, c_2, c_3 and c_4. Then, $-c_2/2$ and $c_1/2$ can be replaced by c_1 and c_2. □