SOLUTIONS OF EXERCISES OF WEEK FOUR

Exercise 1. For each of the following differential equation, write a normal form and its domain.
Also, check whether the function is a solution (Sol.) to the corresponding differential equation (Eq.)

(1) $Sol. : (e^{2x}, (0, 1))$ $\qquad$ $Eq. : 4y''(x) - y(x) = 0$

(2) $Sol. : (\sqrt{1-x}, [0, 1])$ $\qquad$ $Eq. : 2y(x)y'(x) = -1$

(3) $Sol. : (e^{x^2/2}, (-\infty, +\infty))$ $\qquad$ $Eq. : y'(x)/x = y(x)$

(4) $Sol. : (x^2, (-\infty, +\infty))$ $\qquad$ $Eq. : y'(x) = 2\sqrt{y(x)}.$

Solution.

(1) The normal form is $F(x, y(x), y'(x), y''(x)) = 0$ where

$$F : \mathbb{R}^4 \to \mathbb{R}, \quad F(x, y, p_1, p_2) = 4p_2 - y.$$

We have

$$4(e^{2x})'' - e^{2x} = 15e^{2x} \neq 0.$$

Then $(e^{2x}, (0, 1))$ is not a solution.

(2) The normal form is $F(x, y(x), y'(x)) = 0$ where

$$F : \mathbb{R}^3 \to \mathbb{R}, \quad F(x, y, p) = 2py + 1.$$

Since $\sqrt{1-x}$ is not derivable at $x = 1$, $(\sqrt{1-x}, [0, 1])$ is not a solution.

(3) The normal form is $F(x, y(x), y'(x)) = 0$ where

$$F : (\mathbb{R} - \{0\}) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad F(x, y, p) = \frac{p}{x} - y.$$

If $x = 0$, then

$$(0, y(0), y'(0))$$

does not belong to the domain of F. So, it is not a solution.

(4) The function F of the normal form is

$$F : \mathbb{R} \times [0, +\infty) \times \mathbb{R} \to \mathbb{R}, \quad F(x, y, p) = p - 2\sqrt{y}.$$

Then $F(x, y(x), y'(x)) = 0$ if and only if

$$2x = 2|x|$$

which is true only if $x \geq 0$. Then $(x^2, (-\infty, +\infty)$ is not a solution.

□

Date: 2014, October 13.

Exercise 2. Integrate each of the following differential equations

$$y'(x) = y(x)(1 - y(x)) \tag{5}$$

$$y'(x) + 2xy^2(x) = 0. \tag{6}$$

Among the solutions of (5) find at least three solutions with existence interval $\mathbb{R}$. Among the solutions of (6) find at least one solution such that the existence interval is not $\mathbb{R}$.

Solution.

(5) Without integrating the equation, we can find two solutions defined on $(-\infty, +\infty)$, the constants

$$(y_0(x) = 0, (-\infty, +\infty))$$
$$(y_1(x) = 1, (-\infty, +\infty)).$$

We integrate the equation with the separable variables technique. Then, suppose that $y(x)(1 - y(x)) \neq 0$ for every x. Then

$$\frac{y'(x)}{y(x)(1 - y(x))} = 1.$$

That is

$$\left(\frac{1}{y(x)} - \frac{1}{y(x) - 1}\right) y'(x) = 1.$$

Integrating, we obtain

$$\ln |y(x)| - \ln |y(x) - 1| = x + c$$

which we can write

$$\left|\frac{y(x)}{y(x) - 1}\right| = e^c e^x.$$

Now, we need to find an explicit solution. Let us consider the case there $0 < y < 1$. Then

$$\frac{y(x)}{y(x) - 1} = de^x$$

where $d = -e^c$. Then

$$y(x) = -\frac{de^x}{1 - de^x}.$$

Then, if we choose $c = 0$, or $d = -1$, we obtain the third solution on $(-\infty, +\infty)$

$$\left(y(x) = \frac{e^x}{1 + e^x}, (-\infty, +\infty)\right)$$

(6) the constant solution 0 is defined on $(-\infty, +\infty)$. So, in order to find a solution which is not defined on $\mathbb{R}$ we have to integrate the equation. We have

$$\frac{y'}{y^2} = -2x$$

whence

$$-\frac{1}{y} = -x^2 + c.$$

Then

$$y_c(x) = \frac{1}{x^2 - c}.$$

If $c \geq 0$, then the function above is not defined on all the real numbers. If we take $c = 0$, we obtain

$$\left(y_0(x) = \frac{1}{x^2}, (0, +\infty)\right).$$

□

Exercise 3. Let g and f be two derivable Lipschitz functions on the interval $[0, 1]$. Is fg a Lipschitz function?

Solution. First, we check that a Lipschitz function on $[0, 1]$ is bounded. In fact,

$$|f(x)| \leq |f(x) - f(0)| + |f(0)| \leq |f(0)| + L_f|x| \leq |f(0)| + L_f$$

where L_f is the Lipschitz constant of f. Similarly, g is bounded by $L_g + |g(0)|$. Since f and g are derivable,

$$\begin{aligned}|(fg)'(x)| &= |f'g(x) + fg'(x)| \leq |f'g(x)| + |fg'(x)| \\ &\leq L_f(L_g + |g(0)|) + L_g(L_f + |f(0)|).\end{aligned}$$

Since fg has bounded derivative on an interval, it is a Lipschitz function. □

Exercise 4. Let y be a one-variable function which is 1 on the interval $(0, 1)$ and 2 on the interval $(1, 2)$. Is it Lipschitz?

Solution. It is not Lipschitz. In fact, on the sequences

$$x_n := 1 - \frac{1}{2n}, \quad x'_n := 1 + \frac{1}{2n}$$

we have

$$\left|\frac{y(x_n) - y(x'_n)}{x_n - x'_n}\right| = n$$

which goes to infinity as n goes to infinity. □

Exercise 5. Check whether each of the following functions are Lipschitz or locally Lipschitz (if it is locally Lipschitz, write explicitly what is r in $Q_r(x_0, y_0)$)

(7) $$g_1 \colon (0, 1) \times (0, 1) \to \mathbb{R}, \quad g_1(x, y) = \sin(1/x)$$

(8) $$g_2 \colon \mathbb{R} \times [0, 4\pi] \to \mathbb{R}, \quad g_2(x, y) = |\sin y|$$

(9) $$g_3 \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad g_3(x, y) = xy(1 - y)$$

(10) $$g_4 \colon (1, 2) \to \mathbb{R}, \quad g_4(x) = \frac{|x - 1|}{x}.$$

Solution.

(7) Since $\partial_x g$ is not bounded, g is not Lipschitz. However, it is locally Lipschitz. In fact, given (x_0, y_0) in $(0, 1) \times (0, 1)$, we take

$$r := \min\{x_0, 1 - x_0, y_0, 1 - y_0\}/2.$$

Then, $\partial_y g = 0$ and

$$|\partial_x g_1(x, y)| = \left|-\frac{1}{x^2} \sin\frac{1}{x}\right| \leq 2\max\{x_0^{-1}, (1 - x_0)^{-1}\}$$

is bounded

(8) g_2 is not derivable on the domain of definition. However, it is derivable on the intervals $I_k := (k\pi, (k+1)\pi)$ for every $0 \le k \le 3$. On each of these intervals

$$|\partial_y g_2(x,y)| \le 1.$$

Then g_2 is Lipschitz on I_k for every $0 \le k \le 3$. Since g is continuous on $[0, 4\pi]$, it is Lipschitz. Then, is also locally Lipschitz.

(9) $\partial_x g_3 = y(1-y)$ is not bounded on $\mathbb{R}^2$. Then g_3 is not Lipschitz. However, it is locally Lipschitz: given (x_0, y_0), we choose $r = 1$. Then

$$|\partial_x g_3(x,y)| = |y(1-y)| \le (|y_0|+1)(|y_0|+2)$$

and

$$|\partial_y g_3(x,y)| = |x(1-2y)| \le (|x_0|+1)(3+2|y_0|).$$

(10) On the interval $(1,2)$, $x-1>0$. Then

$$g_4(x,y) = \frac{x-1}{x} = 1 - \frac{1}{x}$$

and $\partial_y g_4 = 0$ (bounded) and

$$\partial_x g_4 = \frac{1}{x^2} \le 1.$$

Then g_4 is a Lipschitz function.

□

Exercise 6. Let $(y, (0,1))$ be a solution to the differential equation

$$y'(x) = y(x) \sin y(x)$$

such that $y(0) = \pi/2$. Show that $0 < y(x) < \pi$ for every $0 \le x \le 1$.

Solution. We see that there are two constant solutions

$$(y_0 = 0, (0,1)), \quad (y_1 = \pi, (0,1)).$$

We write the equation as

$$y'(x) = f(y(x))$$

where $f(y) = y \sin y$. The function is locally Lip_y because

$$\partial_y f(x,y) = \sin y + y \cos y$$

is a locally bounded function. Since f it also continuous, it satisfies the hypotheses of the Picard-Lindelöf Theorem.

We claim that $y \ne y_0$ on $(0,1)$. In fact, suppose that there exists x_* in $(0,1)$ such that $y(x_*) = y_0$. By the uniqueness of the Initial Value Problem, we should have $y = 0$ on $(0,1)$. However, this is not possible, because $y(0) = \pi/2$.

Similarly, $y \ne y_1$ on $(0,1)$. In fact, if $y = y_1$ at some point, we had $y = y_1 = \pi$ on $(0,1)$, which, again, contradicts $y(0) = \pi/2$.

Then, for every $x \in (0,1)$, we have $y(x) \ne 0$ and $y(x) \ne \pi$. We show that $0 < y(x) < \pi$: if there exist x_0 such that $y(x_0) > \pi$, then there exists x_1 such that $y(x_1)$ because y and continuous and $y(0) < \pi$. This contradicts the conclusions of the previous paragraph. Similarly, $y > 0$ on $(0,1)$. □