MIDTERM EXAM

Exercise 1. Integrate the differential equation

$$y''(x) = y'(x)^2 + 1$$

with the substitution $y' = z$. Then, find a solution such that $y(0) = 1$ and $y'(0) = 0$.

Solution. We use the substitution $y' = z$. Then

$$z' = z^2 + 1$$

gives

$$\frac{z'}{z^2+1} = 1.$$

Then

$$\arctan z(x) = x + c \Rightarrow z(x) = \tan(x + c). \tag{1}$$

Now we need to solve $y' = z$. Then

$$\left(y^k_{c,d}(x) = -\ln|\cos(x+c)| + d, \left(-\frac{\pi}{2} + k\pi - c, \frac{\pi}{2} + k\pi - c\right)\right) \tag{2}$$

for every k in $\mathbb{Z}$. We need to satisfy the condition $y'(0) = z(0) = 0$. Then, from (1), we have $\tan c = 0$ which implies

$$c = m\pi, \ m \in \mathbb{Z}.$$

From $y(0) = 1$, we obtain

$$1 = -\ln|\cos(c)| + d = -\ln|\cos(m\pi)| + d = d.$$

Then $d = 0$. Among of all the possible solutions in (2), we have to choose the interval which contains 0. Then, a solution is

$$\left(y(x) = -\ln|\cos x| + 1, \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right).$$

□

Exercise 2. Find the constant solutions of the differential equation

$$y'(x) = 2y(x) - \pi \sin y(x).$$

Solution. Constant solutions correspond to the zeroes of the function

$$2s - \pi \sin s.$$

This function has at least three zeroes: $0, \pi/2$ and $-\pi/2$. We show that there are no other zeroes than the ones mentioned above. Since $f(s) := 2s - \pi \sin s$ is an odd function, we only need to check the interval $[0, +\infty)$. If $s > \pi/2$, then

$$|f(s)| \geq |2s| - |\pi \sin s| > \pi - \pi|\sin s| \geq 0.$$

Now, we show that in $(0, \pi/2)$ there are no zeroes. We argue by contradiction: if there exists

$$0 < s_* < \frac{\pi}{2},$$

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then f' should have two zeroes in $(0, s_*)$ and $(s_*, \pi/2)$, respectively. Since

$$f'(s) = 2 - \pi \cos s.$$

then, there are two points $s_1 \neq s_2$ such that

$$\cos s_1 = \cos s_2 = \frac{2}{\pi}.$$

This gives a contradiction, because the function cos is injective on the inverval $[0, \pi/2]$. □

Exercise 3. Consider the following differential equation

$$y(x)e^x(x^2 + 2x) + x^2 e^x (y(x) + 2)y'(x) = 0$$

(a) is it exact?
(b) is there an integrating factor which does not depend on x?
(c) is there an integrating factor which does not depend on y?
(d) find an implicit solution.

Solution.
(a) $M(x, y) = ye^x(x^2 + 2x)$ and $N(x, y) = x^2 e^x (y + 2)$. Then

$$\partial_y M = e^x(x^2 + 2x), \quad \partial_x N = (y + 2)e^x(x^2 + 2x).$$

Since $\partial_y M \neq \partial_x N$, the equation is not exact

(b) since

$$\frac{\partial_x N - \partial_y M}{M} = \frac{(y + 1)e^x(x^2 + 2x)}{yx^2 e^x (x^2 + 2x)} = 1 + \frac{1}{y}$$

there exists and integrating factor which does not depend on x. This is

$$\nu(y) = ye^y$$

(c) since

$$\frac{\partial_y M - \partial_x N}{N} = \frac{(y + 1)e^x(x^2 + 2x)}{x^2 e^x (y + 2)}$$

does not depend only on x, there is no integrating factor depending only on x;

(d) we multiply by ye^y and obtain

$$ye^y M = y^2 e^y e^x (x^2 + 2x) = \partial_x G.$$

Then

$$G(x, y) = x^2 y^2 e^x e^y + c(y).$$

Then

$$\partial_y G(x, y) = x^2(y^2 + 2y)e^x e^y + c'(y) = ye^y N = x^2 e^x y(y + 2)e^y$$

which implies $c' = 0$. Then

$$G(x, y) = x^2 y^2 e^x e^y + c.$$

□

Exercise 4. Make an example of a first order differential equation which is Bernouilli, homogeneous (of degree 3) and non-exact.

Solution. An example given by

$$y'(x)x^3 + y^3(x) = 0.$$

□