SOLUTIONS OF THE EXERCISES OF WEEK THIRTEEN

Exercise 1. Let $\mathbf{Q}$ be the set of rational numbers. Show that there is no open interval I such that $I \neq \emptyset$ and $I \subseteq \mathbf{Q}$.

Solution. If I is an open interval, then $I \approx (-1,2)$. Since $[0,1] \subseteq (-1,2)$, $(-1,2)$ is uncountable, because $[0,1]$ is uncountable. Then I is uncountable, but $\mathbf{Q}$ is countable, so we obtain a contradiction. □

Exercise 2. The set $\mathbf{R} - \mathbf{Q}$ is dense in $\mathbf{R}$.

Solution. Let $a < b \in \mathbf{R}$. If $(a,b) \cap \mathbf{R} - \mathbf{Q} = \emptyset$, then

$$(a,b) \subseteq \mathbf{Q}.$$

We obtain a contradiction, because an uncountable set is contained in a countable set.

Here is another solution:

We showed that there exists $r \in \mathbf{R} - \mathbf{Q}$ such that $r > 0$ and $r^2 = 2$. We also proved that

$$0 < r < 2.$$

Given an open interval $(a,b) \subseteq \mathbf{R}$, with $a < b$, there are rational numbers $a < q_1 < q_2 < b$ because Q is dense. Then

$$(q_1, q_2) \subseteq (a,b).$$

Since r is not rational,

$$q_1 < q + \frac{q_2 - q_1}{2} r < q_2$$

is not rational. □

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