WEEK TEN

2016 February 29, Week 1 - Lecture 1

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Definition 1 (Generalized unions and intersection, from (9.))**.** Given a collection of sets $\mathscr{B} \neq \emptyset$, we define $\cap\mathscr{B}$ and $\cup\mathscr{B}$ as follows:

$$x \in \cup\mathscr{B} \Leftrightarrow \exists A \text{ s.t. } x \in A \in \mathscr{B}$$

$$x \in \cup\mathscr{B} \Leftrightarrow \forall A \in \mathscr{B}(x \in A).$$

Definition 2 (Choice functions, from (10.))**.** Given a set $A \neq \emptyset$, a choice function ϕ is a function $\phi\colon \mathscr{P}(A)^* \to A$ such that $\phi(B) \in B$ for every $B \in \mathscr{P}(A)^*$.

Axiom 1 (Choice Axiom, from (11.))**.** Every non-emptyset A has a choice function.

Proposition 1 (Properties of the quotient set, from (14.))**.** Given an equivalence relation (A, R) the following properties hold:

(i) for every $H \in A/R$, the set H is non-empty
(ii) $\cup A/R = A$

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(iii) given H, G in A/R there holds

$$(H \cap G \neq \emptyset) \Rightarrow (H = G).$$

Proof.
(i). Let $x \in A$ be such that $H = G_x$. From (R), $x \in R_x$, therefore, H is non-empty.
(ii). Since A/R is a collection of subsets of A, we have

$$\cup A/R \subseteq A.$$

Now, given $x \in A$, from (i), we have $x \in G_x$. From

$$x \in G_x \in A/R$$

we have $x \in A/R$.
(iii). Let x and y be such that $H = R_x$ and $G = R_y$. Suppose that $H \cap G$ is non-empty and let z in A be such that $z \in R_x \cap R_y$. Let $w \in R_x$ then. From (S),

$$xRw \Rightarrow wRx.$$

Since $z \in R_x$, we have xRz From (T),

$$(wRx) \wedge (xRz) \Rightarrow wRz.$$

Since $z \in R_y$, we have zRy. Again, from (T) and (S)

$$(wRz) \wedge (zRy) \Rightarrow wRy \Rightarrow yRw \Rightarrow w \in R_y.$$

By switching the role of x and y, we obtain the reversed inclusion $R_y \subseteq R_x$. Then $R_x = R_y$, implying $G = H$. □

Definition 3 (Fully orderings, from (16.)). A partial ordering (A, R), is a full ordering if for every $x, y \in A$ either xRy or yRx.

Axiom 2 (Class Construction Axiom of Zermelo-Fraenkel, from (17.)). Given a sentence $p(x)$ and a set A there exists a *set* $S \subseteq A$ such that $x \in S \Leftrightarrow x \in A \wedge p(x)$.

Theorem 1 (Uniqueness up to isomorphism, from (19.)). Let $(R_1, +, \cdot)$ and $(R_2, +, \cdot)$ two sets satisfying the field, positivity and completeness Axioms. Then there exists a bijective function $g\colon R_1 \to R_2$ such that

$$g(a+b) = g(a) + g(b), \quad g(ab) = g(a)g(b), \quad g(P_1) \subseteq P_2$$

where P_1 and P_2 are the positive sets.

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44. finite sets, page 13
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46. the Bernstein's Lemma, Lemma 1 .

Proposition 2. $(\mathbf{R}, \leq)$ is an ordered set.

Proof. The reflectivity holds, because $x \leq x \Rightarrow x = x$.
(T). Given $x \leq y$ and $y \leq z$, we have

$$(x = y) \vee (y - x \in P) \tag{1}$$

and

$$(y = z) \vee (z - y \in P). \tag{2}$$

We have to show that $x \leq z$, in all the four cases. If equalities hold in (1) and (2), then

$$(x = y) \wedge (y = z) \Rightarrow x = z \Rightarrow x \leq z.$$

If equality holds in (1) but not in (2), then

$$(x = y) \wedge (z - y \in P) \Rightarrow z - x \in P.$$

The case where the equality holds in (2) but not in (1) is similar. Finally, we consider the case where we have two strict inequalities:

$$(y - x \in P) \wedge (z - y \in P) \Rightarrow (z - y) + (y - x) = z - x \in P.$$

We used P1).
(A). Let $x, y \in \mathbf{R}$ be such that $x \leq y \wedge y \leq x$. If equality holds in one of the two inequalities, then, clearly, $x = y$. So, we study only the case $x < y \wedge (y < x)$. We have

$$y - x, -(y - x) \in P$$

which gives a contradiction with P2). □

Definition 4 (Least and greatest element, from (26.))**.** Let M be an upper bound for E; if $M \in E$ we call it greatest element. Similarly, a least element m for E is a lower bound which belongs to E.

Exercise 1 (from (34.))**.** $\mathbf{N} \cap (0,1)$.

Proof. This follows from the fact that $n \geq 1$ for every $n \in \mathbf{N}$, (47.). □

Exercise 2 (from (35.), ex. 9, page 13)**.** Given $n, m \in \mathbf{N}$ such that $n > m$, $n - m \in \mathbf{N}$.

Proof. Here I am writing a slightly different proof from the one I showed you during the lectures. We consider the property $p(n) : n - 1 \in \mathbf{N}$ and define

$$S := \{n \in \mathbf{N} \mid p(n)\} \cup \{1\}$$

and prove that $S = \mathbf{N}$. Clearly, $1 \in S$, by definition. Now, suppose that $n \in S$. Then

$$(n + 1) - 1 = n \in \mathbf{N}.$$

Now we consider the property

$$q(m) : n > m \Rightarrow n - m \in \mathbf{N}$$

and the set $T := \{m \mid q(m)\}$. We proved that $1 \in T$. Now, suppose that $m \in T$. We will show that $m + 1 \in T$. Given $n > m + 1$, clearly $n \neq 1$. Then $n - 1 \in \mathbf{N}$. We have

$$n > m + 1 \Rightarrow n - 1 > m.$$

Since $m \in T$ and $n - 1 \in \mathbf{N}$, $n - (m + 1) \in \mathbf{N}$. $\square$

Proposition 3 (from (37.))**.** Subsets of $\mathbf{N}$ bounded from above have a maximum.

Proof. Let $E \subseteq \mathbf{N}$ be a non-empty subset bounded from above. From the Completeness Axiom there exists $c := \sup(E)$. We prove that $c = \max(E)$. On the contrary, we consider $c - 1$. Since c is the least upper bound, $c - 1$ is not an upper bound. Then there exists $n \in E$ such that

$$c - 1 < n < c.$$

Clearly $n \neq c$ because, otherwise, we would have $c \in E$. Since $n < c$, it is not an upper bound of E. Then, there exists $m \in E$ such that

$$n < m < c.$$

We set $k := m - n$. By Exercise 2, $k \in \mathbf{N}$ and $0 < k < 1$. Therefore, we obtained a contradiction with (34.). $\square$

Definition 5 (Integers, from (39.))**.** The set of integers $\mathbf{Z}$ is defined by the property

$$p(m) : (m = 0) \vee (m \in \mathbf{N}) \vee (-m \in \mathbf{N}).$$

Definition 6 (Definition of $\mathbf{N}_k$, from (43.))**.** We define $\mathbf{N}_k$ the subset of $\mathbf{N}$ of the natural numbers satisfying $1 \leq n \leq k$.

Lemma 1 (Bernstein's Lemma, from (46.))**.** Given two non-empty sets A and B such that there exists $f \colon A \to B$ injective and $g \colon B \to A$ injective, there exists $h \colon A \to B$ bijective.

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47. If $g \colon A \to B$ is bijective, then $A - \{x\} \approx B - \{g(x)\}$, Lemma 2
48. if $k \geq 2$, then $\mathbf{N}_k - \{x\} \approx \mathbf{N}_{k-1}$ for every, $x \in \mathbf{N}_k$, Lemma 3
49. $\mathbf{N}_h \approx \mathbf{N}_k$ if and only if $h = k$, Proposition 4
50. given $B \neq \emptyset$, if $B \subseteq \mathbf{N}$ is not finite, then $B \approx \mathbf{N}$, Theorem 3, page 13
51. $\mathbf{N} \approx \mathbf{N} - \{x\}$ for every $x \in \mathbf{N}$. Then $\mathbf{N}$ is not finite, Proposition 5
52. if $n = ab$, then $n \geq a, b$
53. $\mathbf{N} \times \mathbf{N}$ is countable, Corollary 4, 14
54. the continuum, $\mathbf{R}$, and functional cardinality, $\mathscr{P}(R)$.

Lemma 2 (From (47.))**.** If A and B are non-empty and there exist $g \colon A \to B$ bijective, then $A - \{x\} \approx B - \{g(x)\}$, provided the two sets are non-empty.

Proof. We set $A' := A - \{x\}$ and $B' := B - \{g(x)\}$. We define $g' \colon A' \to B'$ as $g'(a) = a$. Firstly, $g'(a) \in B'$, otherwise, $g'(a) = g(x)$, hence $a = x$ because g is injective; however, this contradicts the assumption $a \in A'$. Clearly, g is an injective function; given $y \in B'$, there exists $x' \in A$ such that $g(x') = y$. We show that $x' \neq x$; on the contrary, $B' \ni y = g(x') = g(x) \notin B'$ we obtain a contradiction. □

Lemma 3 (From (48.)). If $k \geq 2$, then $\mathbf{N}_k - \{x\} \approx \mathbf{N}_{k-1}$ for every $x \in \mathbf{N}_k$.

Proof. We define the function $g \colon \mathbf{N}_k - \{x\} \to \mathbf{N}_{k-1}$ as follows: $g(y) = y$ if $y \leq x - 1$, $g(y) = x + 1$ if $y \geq x + 1$. □

Proposition 4 (From (49.)). Given $h, k \in \mathbf{N}$, $\mathbf{N}_h \approx \mathbf{N}_k$ if and only if $h = k$.

Proof. If $h = k$, then $\mathbf{N}_h = \mathbf{N}_k$. Then $\mathbf{N}_h \approx \mathbf{N}_k$. In order to prove the converse implication, we apply the Mathematical Induction Principle to the set

$$S := \{h \in \mathbf{N} \mid \forall k(\mathbf{N}_h \approx \mathbf{N}_k \Rightarrow h = k)\}.$$

(i) $1 \in S$. If $\mathbf{N}_1 \approx \mathbf{N}_k$, then there exists a bijective function $g \colon \mathbf{N}_1 \to \mathbf{N}_k$. Thus, g is surjective and

$$\mathbf{N}_k = g(\mathbf{N}_1) = \{g(1)\}.$$

Since $1, y \in \mathbf{N}_k$, we have $1 = y = g(1)$.

(ii) $h \in S \Rightarrow h + 1 \in S$. Suppose that $\mathbf{N}_{h+1} \approx \mathbf{N}_k$. Firstly, we notice that $k \geq 2$. In fact, if $k = 1$, we have

$$\mathbf{N}_{h+1} \approx \mathbf{N}_1.$$

We can, then, apply the case $1 \in S$ and conclude that $h + 1 = 1$ which contradicts the assumption that $h \in \mathbf{N}$. There exists a bijective function $g \colon \mathbf{N}_{h+1} \to \mathbf{N}_k$. Since $k \geq 2$, we can apply Lemma 2 and obtain

$$\mathbf{N}_h = \mathbf{N}_{h+1} - \{h + 1\} \approx \mathbf{N}_k - \{g(h + 1)\}$$

In fact, both sets are non-empty, because $k \geq 2$. Then, from Lemma 3

$$\mathbf{N}_k - \{g(h + 1)\} \approx \mathbf{N}_{k-1}.$$

Then $\mathbf{N}_h \approx \mathbf{N}_{k-1}$. Since $h \in S$, we have $h = k - 1$. Then $h + 1 = k$. □

Proposition 5 (From (51.)). $\mathbf{N} \approx \mathbf{N} - \{x\}$ for every $x \in \mathbf{N}$. Then $\mathbf{N}$ is not finite.

Proof. The bijective function is defined as $g(k) = k$ if $k \leq x - 1$ and $g(k) = k + 1$ if $k \geq x$. Now, if $\mathbf{N}$ was finite, then $\mathbf{N} \approx \mathbf{N}_k$ for some $k \geq 1$. Since $1, 2 \in \mathbf{N}$ we can suppose that $\mathbf{N} - \{x\}$ is non-empty. Then, from Lemma 2 and Lemma 3, we obtain

$$\mathbf{N}_k \approx \mathbf{N} \approx \mathbf{N} - \{x\} \approx \mathbf{N}_k - \{y\} \approx \mathbf{N}_{k-1}$$

which implies $k = k - 1$, from Proposition 4 and we obtain a contradiction. □

2016 March 17, Week 3 - Lecture 2

55. $\mathbf{Q}$ is countable, Proposition 6
56. non-degenerate intervals in $\mathbf{R}$ are uncountable, Theorem 7, page 15
57. $I_r(x) := (x - r, x + r)$
58. open sets, Definition page 16
59. union and finite intersections of open sets are open, Proposition 8
60. closure points, Definition, page 17
61. closed sets, page 17
62. convex sets, Definition 7
63. intervals are convex, Proposition 7

64. bounded convex sets are intervals, Proposition 8.

Proposition 6 (From (55.)). $\mathbf{Q}$ is countable.

Proof. Given $q \in Q$, we define

$$E_q := \{q \in \mathbf{N} \mid qx \in \mathbf{Z}\}.$$

Since q is a rational number, $E_q \neq \emptyset$. Then, by the Well Ordering Theorem, the set has a minimum. We define

$$h\colon Q \to \mathbf{Z} \times \mathbf{N}, \quad h(q) := (q \min(E_q), \min(E_q)).$$

This function is injective: given q, q' such that $h(q) = h(q')$, we have

$$q \min(E_q) = q' \min(E_{q'}), \quad \min(E_q) = \min(E_{q'}).$$

If we substitute the equality on the right in the left equality, we obtain $q = q'$. Then, we have a chain of injective functions

$$\mathbf{Q} \to \mathbf{Z} \times \mathbf{N} \to \mathbf{N} \times \mathbf{N} \to \mathbf{N}$$

which is injective. Thus, by (50.), $\mathbf{Q}$ is countable. □

Definition 7 (Convex sets, from (62.)). A subset $S \subseteq \mathbf{R}$ is convex if for every $x \leq y \in S$, there holds $[x, y] \subseteq S$.

Proposition 7 (Intervals are convex, from (63.)). Intervals are convex.

Proof. Let $[a, b]$ be a closed bounded interval. Given $x, y \in [a, b]$, we have $a \leq x \leq b$ and $a \leq y \leq b$. Then if $x \leq z \leq y$, we have $a \leq x \leq z \leq y \leq b$, so $z \in [a, b]$. □

Proposition 8 (Convex sets are interval, from (64.)). Convex sets are intervals.

Proof. Now, suppose that S is convex. We divide the proof into four cases:
(1), S is bounded. By the Completeness Axiom there are $a := \inf(S)$ and $b := \sup(S)$. Clearly, $S \subseteq [a, b]$. Now, we show that $(a, b) \subseteq S$. In fact, let x be an element of (a, b). Then

$$a < x.$$

Since a is the g.l.b., x is not a lower bound of S. Therefore, there exists $x' \in S$ such that $x' < x$. We also have $x < b$. Since b is the l.u.b., x is not an upper bounded for S. Then, there exists $x'' \in S$ such that $x < x''$. Since S is convex, $[x', x''] \subseteq S$. Since $x' < x < x''$, also $x \in S$. Since $a \leq x'$ and $x'' \leq b$, then the set S satisfies

$$(a, b) \subseteq S \subseteq [a, b].$$

There are only four sets satisfiying the inclusions above:

$$[a, b], \quad (a, b], \quad [a, b), \quad (a, b).$$

(2). S is unbounded from below and bounded from above. We define

$$b := \sup(S).$$

We can prove that

$$(-\infty, b) \subseteq S \subseteq (-\infty, b]. \tag{3}$$

Suppose that $x < b$. Then x is not the l.u.b. Then, there exists $x' \in S$ such that $x < x'$. Since S is not bounded from below, $-(|x| + 1)$ is not a lower bound. Then, there exists $x'' \in S$ such that

$$x'' < -(|x| + 1) < x < x'.$$

Since S is convex,

$$x \in [x', x''] \subseteq S.$$

Then $x \in S$. Sets satisfying (3) are only

$$(-\infty, b), \quad (-\infty, b].$$

The cases (3) and (4) where S is unbounded from above and bounded from below, or S is unbounded from below and from above, are similar to the case (2). □

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65. Generalized intersection of convex sets, Proposition 9
66. generalized union of convex sets, Proposition 10
67. convex sets are intervals, Proposition 11
68. open sets are countable unions of pairwise disjoint intervals, Theorem 2
69. open covers, page 18
70. compact sets, Definition 8 .

Proposition 9 (From (65.))**.** If $\mathscr{C}$ is a collection of convex set such that $\cap\mathscr{C} \neq \emptyset$, then $\cap\mathscr{C}$ is convex.

Proof. Let x, y be two elements of $\cap\mathscr{C}$. Then $x, y \in C$ for every $C \in \cap\mathscr{C}$. Since C is convex, $[x, y] \subseteq C$ for every $C \in \cap\mathscr{C}$, which implies $[x, y] \subseteq \cap\mathscr{C}$. □

Proposition 10 (From (66.))**.** If $\mathscr{D}$ is a collection of convex set such that $\cap\mathscr{D} \neq \emptyset$, then $\cup\mathscr{D}$ is convex.

Proof. Let x_0 be an element of $\cap\mathscr{D}$. Let $x, y \in D := \cup\mathscr{D}$ be two elements. Then, there are D_1 and D_2 such that $x \in D_1$ and $y \in D_2$. Here we consider different cases:
$x_0 \leq x \leq y$. Then $[x, y] \subseteq [x_0, y] \subseteq D_2 \subseteq D$ because D_2 is convex. If $x \leq y \leq x_0$, then $[x, y] \subseteq [x, x_0] \subseteq D_1 \subseteq D$ because D_1 is convex. Finally, if $x \leq x_0 \leq y$, we have

$$[x, x_0] \subseteq D_1 \text{ and } [x_0, y] \subseteq D_2 \Rightarrow [x, y] \subseteq D_1 \cup D_2.$$

□

Proposition 11 (From (67.))**.** Convex sets are intervals.

Proof. Let C be a convex non-empty set and let $x_0 \in C$ be an element of C. We define $C_n := C \cap (-n+x_0, n+x_0)$ for every natural number $n \geq 1$. We also define the collection of intervals

$$\mathscr{C} := \{C_n \mid n \geq 1\}, \quad C = \cup\mathscr{C}.$$

From Proposition 9, C_n is convex. Moreover, $\cap\mathscr{C} = (-1 + x_0, 1 + x_0) \neq \emptyset$. By Proposition 10, $\cup\mathscr{C}$ is convex. □

Theorem 2 (From (68.))**.** Let Ω be an open non-empty set of $\mathbf{R}$. Then Ω is countable union of open intervals which are disjoint from each other.

Proof. Let x be an element of Ω. Since Ω is open, there exists $r > 0$ such that $I_r(x) \subseteq \Omega$. We define the collection

$$\mathscr{A}_x := \{J \text{ open interval } \mid x \in J \subseteq \Omega\}. \tag{4}$$

Clearly, $\mathscr{A}_x$ is non-empty, because $I_r(x) \in \mathscr{A}_x$. We also define $J_x := \cup\mathscr{A}_x$ The sets J_x satisfy some properties that we list below

(i) $x \in J_x$
(ii) J_x is an open set
(iii) J_x is an interval
(iv) given $x, y \in \Omega$, either $J_x = J_y$ or $J_x \cap J_y = \emptyset$

(i). Since $I_r(x) \in \mathscr{A}_x$ and $x \in I_r(x)$, then $x \in J_x$ by definition of generalized union. (ii) J_x is open because is the union of a collection of open sets, namely $\mathscr{A}_x$. (iii) J_x is an interval: $\mathscr{A}_x$ is a collection of intervals, that is, convex sets. Each of these sets contain x, from (i). Thus, $\cap\mathscr{A}_x \neq \emptyset$. From Proposition 10, J_x is a convex set. From Proposition 11, J_x is an interval.
(iv). Let us suppose that $J_x \cap J_y \neq \emptyset$. Then there exists z in the intersection. Therefore, by Proposition 10 and Proposition 11, $J_x \cup J_y$ is an interval. It is open, because is the union of two open sets. Since $x \in J_x \cup J_y$, there holds.

$$J_x \cup J_y \in \mathscr{A}_x.$$

Then

$$J_x \cup J_y \subseteq \cup\mathscr{A}_x = J_x$$

therefore $J_y \subseteq J_x$. Similarly, from $y \in J_x \cup J_y$, we can show that $J_y \subseteq J_x$. Thus, $J_x = J_y$. We define the collection G as follows:

$$J \in G \Leftrightarrow \exists x \in \Omega \text{ s.t. } H = J_x.$$

From (i), $\cup G = \Omega$. We claim that G is countable. Since $\mathbf{Q}$ is dense in $\mathbf{R}$, for every $J \in G$, the set $J \cap \mathbf{Q}$ is non-empty. We define the following function

$$f\colon G \to \mathbf{Q}, \quad f(J) = \phi(J \cap Q)$$

where ϕ is a Choice Function for $\mathbf{R}$. This function is injective. In fact, given $J_1, J_2 \in G$, there holds

$$f(J_1) = f(J_2) \Rightarrow \phi(J_1 \cap Q) = \phi(J_2 \cap Q).$$

We define $w := \phi(J_1 \cap Q) = \phi(J_2 \cap Q)$. Since ϕ is a choice function,

$$w \in (J_1 \cap Q) \cap (J_2 \cap Q) \subseteq J_1 \cap J_2.$$

From (iv), $J_1 = J_2$. □

Definition 8 (Compact sets, from (70.)). A non-empty set $E \subseteq \mathbf{R}$ is compact if for every open cover $\mathscr{U}$ there exists a finite sub-cover $\mathscr{U}' \subseteq \mathscr{U}$.

2016, March 24 - Week 4, Lecture 2

71. Solutions of the exercises of Week Three
72. Heine-Borel Theorem, page 18.

Week 5, Lecture 1 - 2016, March 28

73. σ-algebras, Definition 9
74. σ-algebras generated by collections, Definition 10 and Proposition 12
75. the Borel's σ-algebra, Definition, page 20
76. G_δ sets and F_σ sets, page 20
77. properties of measures $m\colon \mathscr{B} \to [0, +\infty]$

77.1. translation invariance $m(A + y) = m(A)$

77.2. finite additivity, $m(A \cup B) = m(A) + m(B)$

77.3. σ-additivity: if (A_n) is a disjoint countable collection

$$m(\bigcup_{n=1}^{+\infty} A_n) = \sum_{n=1}^{+\infty} m(A_n)$$

77.4. σ-sub-additivity: if (A_n) is a countable collection

$$m(\bigcup_{n=1}^{+\infty} A_n) \leq \sum_{n=1}^{+\infty} m(A_n)$$

77.5. $m^*(I) = \ell(I)$

78. length of intervals

78.1. bounded intervals: $I \in \{(a,b), [a,b), (a,b], [a,b]\}$, $\quad \ell(I) := b - a$

78.2. unbounded intervals: $\ell(I) = \infty$

79. length of collection of intervals, Definition 11

80. the Lebesgue outer measure, §2.2, page 31.

Definition 9 (σ-algebras, from (73.))**.** A collection of sets $\mathscr{B}$ is a σ-algebra if it satisfies the following properties:

(i) $\emptyset \in \mathscr{B}$
(ii) $E \in \mathscr{B} \Rightarrow E^c \in \mathscr{B}$
(iii) given a countable collection $G \subseteq \mathscr{B}$, there holds $\cup G \in \mathscr{B}$.

Definition 10 (Length of collection, from (74.))**.** Given a collection $A \subseteq \mathscr{P}(\mathbf{R})$, we define

$$\mathscr{F}_A = \{\mathscr{B} \mid (A \subseteq \mathscr{B}) \wedge (\mathscr{B} \text{ is a } \sigma\text{-algebra}\} \subseteq \mathscr{P}(\mathscr{P}(\mathbf{R})).$$

We define $\mathscr{B}(A) := \cap \mathscr{F}_A$.

Proposition 12 (From (74.))**.** $\mathscr{B}(A)$ is a σ-algebra.

Proof.

(i) For all $\mathscr{B}$ we have $\emptyset \in \mathscr{B} \in \mathscr{F}_A$. Then $\emptyset \in \mathscr{B}(A)$
(ii) $E \in \mathscr{B}(A)$ implies $E \in \mathscr{B} \in \mathscr{F}_A$ for every $\mathscr{B}$. Then $E^c \in \mathscr{B}$, hence $E^c \in \mathscr{F}_A$
(iii) let $G \subseteq \mathscr{B}(A)$ be a countable collection. Then, $G \subseteq \mathscr{B}$ for every $\mathscr{B} \in \mathscr{F}_A$. Then $\mathscr{B} \subseteq \cap \mathscr{F}_A$.

□

Definition 11 (Length of collection, from (79.))**.** Given a countable collection of intervals J, we define its length as $L(J) := \sum_{n=1}^{+\infty} \ell(I_n)$, where $I_n \in J$.

Week 10, Lecture 1 - 2016, May 2

81. Monotonicity of the outer measure, page 31

82. the outer-measure of a countable set is zero, Example in page 31

83. given two bounded intervals I_1, I_2, $\ell(I_1 \cup I_2) \leq \ell(I_1) + \ell(I_2)$

84. Proposition 13

85. the outer measure of an interval is equal to its length, Theorem 3.

Proposition 13 (From (84.))**.** If $[a,b] \subseteq \cup J$ there J is a disjoint collection of open interval, then there exists $I \in J$ such that $[a,b] \subseteq I$.

Proof. Suppose that there are two intervals $I_1 \neq I_2$ such that $a \in I_1$ and $b \in I_2$. Without loss of generality we can suppose that I_1 is bounded from above, I_2 is bounded from below and $\sup(I_1) = s \leq S = \inf(I_2)$. Since all the intervals are open, $s \notin \cup J$. Since $a \in I_1$, we have $a < s$. Since $b \in I_2$, we also have $s < b$. Therefore, $s \in [a,b] \subseteq \cup J$ gives a contradiction. □

Theorem 3 (From (85.))**.** If A is an interval, then $m^*(A) = \ell(A)$.

Proof. Firstly, we consider bounded intervals and, in particular, the closed bounded interval $[a,b]$. For every natural number n, we have the collection

$$J_n := \{(a - 1/2n, b + 1/2n)\}, \quad [a,b] \subseteq \cup J_n.$$

Then $m^*([a,b]) \leq L(J_n) = b-a+1/n$. The last inequality holds for every $n \in \mathbf{N}$. Then $m^*([a,b]) \leq b-a$.

We prove the converse inequality. Given $n \in \mathbf{N}$, there exists J_n such that $[a,b] \subseteq J_n$ and $L(J_n) \leq m^*(A) + 1/n$. By the Heine-Borel's theorem, there exists a finite subcover J'_n. We claim that $L(J'_n) \geq b-a$ and prove the claim by induction on $k := \#J'_n$. If $k = 1$, then J'_n contains exactly one interval, namely, $I = (c,d)$. Since $[a,b] \subseteq (c,d)$ we have

$$c < a \leq b < d.$$

Then $L(J'_n) = \ell(I) = d - c > b - c > b - a = \ell([a,b])$. Now, we prove that $k \Rightarrow k+1$. We consider the two different cases:

First case. J'_n is a disjoint collection of open intervals. Then, by Proposition 13, there exists $I \in J'_n$ such that $[a,b] \subseteq I$. Then

$$\ell([a,b]) \leq \ell(I) \leq L(J'_n).$$

Second case. There are two intervals $I', I'' \in J'_n$ such that $I' \neq I''$ and $I' \cap I'' \neq \emptyset$. Then we define $\tilde{I} := I' \cup I''$ which is interval because the intersection is non-empty. We define

$$J''_n := J'_n \cup \{\tilde{I}\} - \{I', I''\}$$

which is an open cover of $[a,b]$ and $\#J''_n = k$. Then, by the inductive hypothesis,

$$b - a \leq L(J''_n) \leq L(J'_n)$$

The second inequality follows from $\ell(\tilde{I}) \leq \ell(I') + \ell(I'')$. This settles the second case. Finally,

$$b - a \leq L(J'_n) \leq L(J_n) \leq m_*([a,b]) + \frac{1}{n}.$$

Taking the limit, we obtain $b - a \leq m_*([a,b])$.

Other bounded intervals. Given $n \geq 1$, we consider the set $[a + 1/2n, b - 1/2n]$. Then $(a,b) \supseteq [a + 1/2n, b - 1/2n]$. Since the outer measure is monotone, from the inclusions

$$[a,b] \supseteq [a,b), (a,b] \supseteq (a,b) \supseteq [a + 1/2n, b - 1/2n].$$

We obtain

$$m^*([a,b]) \geq m^*([a,b)), m^*((a,b]) \supseteq m^*((a,b)) \supseteq m^*([a + 1/2n, b - 1/2n])$$

then

$$b - a \geq m^*([a,b)), m^*((a,b]) \geq m^*((a,b)) \geq b - a - 1/n.$$

Taking the limit, we obtain

$$b - a = m^*([a,b)) = m^*((a,b]) = m^*((a,b))$$

which is equal to the length of each of those intervals.

Unbounded intervals. We use the monotonicity property of the outer-measure. From the inclusions

$$[a, +\infty) \supseteq (a, +\infty) \supseteq (a, n]$$

we obtain that $m^*((a,+\infty)) \geq n - a$ for every $n \in \mathbf{N}$. Then $m^*([a,+\infty)) = m^*((a,+\infty)) = \infty$.

$$(-\infty, b] \supseteq (-\infty, b) \supseteq (-n, b].$$

Then $m^*((-\infty, b) \geq b + n$. Then $m^*((-\infty,b)) = m^*((-\infty,b]) = \infty$. Finally,

$$(\forall n \in \mathbf{N}): \ \mathbf{R} \supseteq (-n/2, n/2) \Rightarrow m^*(\mathbf{R}) \geq n$$

which implies that $m^*(\mathbf{R}) = \infty$. □