CONTENT OF THE LECTURE OF MAY 27TH, 2014

Notation 1. Suppose that $(A, \leq)$ is a p.o.c (partially ordered class) and $B \subseteq A$ is a non-empty subclass. Then

(i) if B has a least element, we denote it with $\min(B)$ ("minimum of B")
(ii) if B has a greatest element, we denote it with $\max(B)$ ("maximum of B")

Definition 1 (Upper bound)**.** $(A, \leq)$ p.o.c , $B \subseteq A$ non-empty.

$u \in A$ is an *upper bound* (u.b.) of B if

$$x \leq u (\forall x \in B).$$

Example 1. $A = \{x, y, z\}$, $B = \{x, y\}$ and

$$R = id_A \cup \{(x, y), (y, z), (x, z)\}$$

y and z are upper bounds of B.

Example 2 (Upper bounds may not exist)**.** $(A, \leq) = (\mathbf{N}, \leq)$. If $B = 2\mathbf{N}$, then B does not have upper bounds. On the contrary,

$$x \leq u (\forall x \in 2\mathbf{N}).$$

Then, $2u \in 2\mathbf{N} \Rightarrow 2u \leq u$ which is false.

Definition 2 (Classes bounded from above)**.** $B \subseteq A$ is bounded from above, if there exists an u.b. for B in A.

Notation 2. Given, $B \subseteq A$, $u(B)$ is the class of upper bounds of B in A.

Then, B bounded from above implies $u(B) \neq \emptyset$.

Definition 3 (Supremum)**.** Let $B \subseteq A$ be such that $u(B) \neq \emptyset$. If $u(B)$ has the least element, then $\min(u(B))$ is called *least upper bound* (l.u.b) of B in A or *supremum* of B in A.

Notation 3 (l.u.b.)**.** For the l.u.b., the notation $\sup_A(B)$ is used.

Example 3. In Example 1, $u(B) = \{y, z\}$ and $\min(u(B)) = \sup_A B = y$.

Example 4 (Classes bounded from above without a l.u.b.)**.** $A = \{a, b, c, d, u_1, u_2\}$, $B = \{a, b, c, d\}$ and

$$R = id_A \cup \{(a, c), (b, d), (a, u_1), (a, u_2), (b, u_1), (b, u_2), (c, u_1), (c, u_2)\}$$

$u(B) = \{u_1, u_2\}$ but $\nexists \min(u(B))$ because u_1 and u_2 are not comparable.

Definition 4. $(A, \leq)$ p.o.c , $B \subseteq A$

(i) $\ell \in A$ is a lower bound (l.b.) of B if $\ell \leq x (\forall x \in B)$
(ii) B is bounded from below if it has a lower bound
(iii) $\lambda(B)$ is the class of lower bounds of B ("lambda of B")
(iv) if $\lambda(B) \neq \emptyset$ and $\max(\lambda(B))$ exists, it is called *infimum* of B in A and the notation $\inf_A B$ is used.

Example 5. In Example 1, instead of (A, R), we consider (A, R^{-1}). Then $B = \{x, y\}$ is bounded from below.

Example 6. In Example 4, we consider (A, R^{-1}), then B is bounded from below but $\nexists \inf_A(B)$.

Given a $(A, \leq)$ p.o.c , we consider

$$S := \{B \subseteq A \mid B \text{ is a chain of } A\}$$

and the order relation

$$B_1 \leq B_2 \Leftrightarrow B_1 \subseteq B_2.$$

Proposition 1. Every chain in $(S, \subseteq)$ has a supremum.

Proof. C chain.

We define

$$m := \cup C \subseteq A.$$

We claim that

(1) $m \in S$
(2) $m \in u(C)$
(3) $m = \sup_S C$.

Proof of the claim:

(1) m is a chain: $x, y \in m$

$$x \in m \Rightarrow \exists B_1 \in C \cdot \ni \cdot x \in B_1$$

and

$$y \in m \Rightarrow \exists B_2 \in C \cdot \ni \cdot y \in B_2.$$

C is a chain. Then $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. For instance, if $B_1 \subseteq B_2$

$$x, y \in B_2.$$

B_2 is a chain implies x is comparable to y.

(2) $m \in u(C) \Leftrightarrow B \subseteq m (\forall B \in C)$, that is $B \subseteq \cup C (\forall B \in C)$. This follows from the definition of generalized union.

(3) let u be an u.b. of C. We claim that $m \leq u$.

$$\forall B \in C (B \subseteq u) \Rightarrow \cup C \subseteq u \Rightarrow m \subseteq u.$$

□

Definition 5 (Well-ordered class)**.** A $(A, \leq)$ p.o.c is a *well-ordered class* (w.o.c) if and only min(B) exists for every $B \subseteq A$ non-empty.

Example 7 (A w.o.c)**.** In Example 1, we have a w.o.c. For instance

$$\min(A) = x, \quad \min(\{x, y\}) = x, \quad \min(\{x, z\}) = x \ldots$$

Example 8 (A p.o.c. which is not a w.o.c.)**.** $A = \{x, y, z, t\}$ and

$$R = id_A \cup \{(z, y), (z, x), (t, y), (t, x), (y, x)\}.$$

$\nexists \min(\{z, t\})$ and $\nexists \min(\{z, t, y\})$.

Example 9 (A f.o.c which is not a w.o.c)**.** $(\mathbf{Z}, \leq)$, relative integers. Is a fully ordered class. However, it is not a w.o.c

$$B := 2\mathbf{Z} \subseteq \mathbf{Z}.$$

B is not bounded from below.

On the contrary,

$$\ell \leq x (\forall x \in 2\mathbf{Z})$$

the integer

$$-2(|\ell| + 1) \in 2\mathbf{Z}$$

then

$$\ell \leq -2(|\ell| + 1)$$

which is false.

F.o.c. are not w.o.c, however

Proposition 2 (W.o.c are f.o.c)**.** $(A, \leq)$ p.o.c . Then $(A, \leq)$ w.o.c $\Rightarrow$ f.o.c .

Proof. $x, y \in A \Rightarrow x$ is comparable to y.

$$B := \{x, y\}$$

A w.o.c $\Rightarrow \exists \min(B)$.

$\min(B) = x \Rightarrow x \leq y$

$\min(B) = y \Rightarrow y \leq x$. □