CONTENT OF THE LECTURE OF MAY 30TH, 2014

Part 1. Partially Ordered Classes

Theorem 1 (Hausdörff Maximum Principle)**.** Given a p.o.c $(A, \leq)$, if A is a set, then there exists a maximal chain.

Definition 1 (Inductive property)**.** A p.o.c $(A, \leq)$ has the inductive property if every chain has a l.u.b (least upper bound).

Example 1. Given an order relation $(A, \leq)$, we can define

$$S := \{B \subseteq A \mid B \text{ is a chain }\}.$$

In S, there is an order relation given by $B_1 \leq B_2$ if and only $B_1 \subseteq B_2$. Then $(S, \leq)$ has the inductive property. In fact, in Proposition 1, we proved that if $C \subseteq S$ is a chain, then

$$\sup_S(C) = \cup C.$$

Example 2. $(\mathbf{N}, \leq)$ does not have the inductive property: for instance, $\mathbf{N}$ and $2\mathbf{N}$ (even numbers) are chains, but have no upper bound (check Example 2).

Theorem 2 (Zorn's Lemma)**.** A partially ordered set $(A, \leq)$ with the inductive property has a maximal element.

Proof. By the Hausdörff Maximal Principle, there exists a maximal chain, B.

Since $(A, \leq)$ has the inductive property, there exists $u := \sup_A(B)$.

We claim that $u \in B$ and is a maximal element of A, that is

$$\forall y \in A (u \leq y \Rightarrow u = y).$$

Define $C := B \cup \{y\}$. We show that C is a chain.

Given $x_1, x_2 \in C$, we have four different cases:

(i) $x_1, x_2 \in B$. Since B is a chain, x_1 is comparable to x_2.
(ii) $x_1 \in B, x_2 \in \{y\}$. Then $x_2 = y$. Since u is an upper bound of B and $u \leq y$, we have $x_1 \leq u \leq y = x_2$
(iii) $x_1, x_2 \in \{y\} \Rightarrow x_1 = x_2$. Then the two elements are comparable because they are equal
(iv) the last case $x_2 \in B$ and $x_1 \in \{y\}$ is similar to the second case.

Then C is a chain and $B \subseteq C$. Since B is a maximal chain, we have $B = C$. Then

$$y \in B.$$

Since u is an upper bound of B, $y \leq u$ which, together with $u \leq y$, give $u = y$. □

Part 2. Natural numbers

Definition 2. Given a set y, we call $y^+ := y \cup \{y\}$ the *successor set of* y.

Example 3.

$\varnothing^+ = \varnothing \cup \{\varnothing\} = \{\varnothing\} = \{0\} = 1.$

$1^+ = 1 \cup \{1\} = \{0\} \cup \{1\} = \{0, 1\} = 2$. That is $0^{++} = 2$.

Clearly, y^+ can be taken as a the definition of "$y + 1$" by means of sets.

Part 3. Solutions of exercises 1 and 3 of Week Thirteen

Exercise 1. Prove, by showing an example $(A, \leq)$, that even when there is only one maximal element m, it is not the greatest element.

Solution. Let $(\mathbf{N}, R)$ be the usual partial ordering on the set of natural numbers.

We consider the set $\omega = \mathbf{N} \cup \{0\}$ and $T := R \cup \{(0, 0), (1, 0)\}$. The order relation (ω, T) has a unique maximal element, namely 0.

In fact, $0 \leq y \Rightarrow (0, y) \in T$. Since $(0, y) \notin R$, there holds

$$(0, y) \in \{(0, 0), (1, 0)\}.$$

Then $(0, y) = (0, 0)$ (because $0 \neq 1$). Then $y = 0$. □

Exercise 2. Express in terms of graph inclusions the statement: (A, R) is a fully ordered class (for instance, the symmetry is expressed in terms of graphs inclusions by the property $id_A \subseteq R$).

Solution. (A, R) is a fully ordered class if and only if $A \times A \subseteq R \cup R^{-1}$. □