EXERCISES OF WEEK ELEVEN

Exercise 1.

1. Given a function $f\colon A \to B$ and $C_1, C_2 \subseteq A$ and $D_1, D_2 \subseteq B$, show that

$$\bar{f}(C_1 \cup C_2) = \bar{f}(C_1) \cup \bar{f}(C_2)$$

and

$$\bar{f}(C_1 \cap C_2) \subseteq \bar{f}(C_1) \cap \bar{f}(C_2)$$

2. show that in some case the equality does not hold. That is, there are $f, A, B, C_1, C_2 \subseteq A$ such that $\bar{f}(C_1 \cap C_2) \neq \bar{f}(C_1) \cap \bar{f}(C_2)$

3. let $C \subseteq A$ be non-empty. Then $\bar{f}(C) \neq \emptyset$

4. $\bar{\bar{f}}(D_1 \cup D_2) = \bar{\bar{f}}(D_1) \cup \bar{\bar{f}}(D_2)$

5. $\bar{\bar{f}}(D_1 \cap D_2) = \bar{\bar{f}}(D_1) \cap \bar{\bar{f}}(D_2)$

Solution.

1. Since $C_1, C_2 \subseteq C_1 \cup C_2$, we have $\bar{f}(C_1), \bar{f}(C_2) \subseteq \bar{f}(C_1) \cup \bar{f}(C_2)$. Then

$$\bar{f}(C_1) \cup \bar{f}(C_2) \subseteq \bar{f}(C_1 \cup C_2).$$

Let $y \in \bar{f}(C_1 \cup C_2)$. Then, there exists $x \in C_1 \cup C_2$ such that $f(x) = y$. If $x \in C_1$, then $y \in \bar{f}(C_1)$. Otherwise, $y \in \bar{f}(C_2)$

2. let $y \in \bar{f}(C_1 \cap C_2)$. Then, there exists $x \in C_1 \cap C_2$ such that $f(x) = y$; $x \in C_1 \cap C_2 \Rightarrow x \in C_1$ which implies $y = f(x) \in \bar{f}(C_1)$. Since $x \in C_2$, $y \in \bar{f}(C_2)$. The equality does not hold, in general. If we choose

$$A = 2, \quad B = 1, \quad f := \{(0,0),(1,0)\}, \quad C_1 = \{0\}, \quad C_2 = \{1\}$$

then $C_1 \cap C_2 = \emptyset$ and $\bar{f}(\emptyset) = \emptyset$. However,

$$\bar{f}(C_1) = \bar{f}(C_2) = \{0\}$$

which implies $\bar{f}(C_1) \cap \bar{f}(C_2) = \{0\}$

3. since $C \neq \emptyset$ there exists $x \in C$. Then $f(x) \in \bar{f}(C)$. Then $\bar{f}(C) \neq \emptyset$

4.

$$\begin{aligned} x \in \bar{\bar{f}}(D_1 \cup D_2) &\Leftrightarrow f(x) \in D_1 \cup D_2 \Leftrightarrow (f(x) \in D_1) \vee (f(x) \in D_2) \\ &\Leftrightarrow (x \in \bar{\bar{f}}(D_1)) \vee (x \in \bar{\bar{f}}(D_2)) \Leftrightarrow x \in \bar{\bar{f}}(D_1) \cup \bar{\bar{f}}(D_2) \end{aligned}$$

5.

$$\begin{aligned} x \in \bar{\bar{f}}(D_1 \cap D_2) &\Leftrightarrow f(x) \in D_1 \cap D_2 \Leftrightarrow (f(x) \in D_1) \wedge (f(x) \in D_2) \\ &\Leftrightarrow (x \in \bar{\bar{f}}(D_1)) \wedge (x \in \bar{\bar{f}}(D_2)) \Leftrightarrow x \in \bar{\bar{f}}(D_1) \cap \bar{\bar{f}}(D_2) \end{aligned}$$

□

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Exercise 2. Let A be a set and $f\colon A \to A$ be an invertible function. Prove that there exists a function g such that

$$f \circ g = \emptyset = g \circ f$$

Solution. Since A is a set, $A' \neq \emptyset$. We define $g := id_{A'}$. Then $f \circ g = g \circ f = \emptyset$ because

$$\operatorname{ran}(g) \cap \operatorname{dom}(f) = \operatorname{ran}(f) \cap \operatorname{dom}(g) = \emptyset.$$

□

Exercise 3. Show that there are classes A, B such that $\cup A \subseteq \cup B$ and $A \not\subseteq B$.

Solution. Let $A = \{\{0,1\}\}$ and $B = \{\{0\},\{1\}\}$. Then

$$\cup A = \cup B = \{0,1\}$$

but $A \not\subseteq B$. □