Orbitally stable coupled standing waves for a coupled non-linear Klein–Gordon equation

Garrisi Daniele

Pohang Mathematics Institute, POSTECH

The 4th MSJ-SI
Nonlinear dynamics in Partial Differential Equations
Kyushu University, September 14, 2011

The coupled non-linear Klein-Gordon equation

$$\begin{aligned} \partial_{tt} v_1 - \Delta_x v_1 + m_1^2 v_1 + \partial_{v_1} F(v) = 0 \\ \partial_{tt} v_2 - \Delta_x v_2 + m_2^2 v_2 + \partial_{v_2} F(v) = 0 \end{aligned} \tag{CNLKG}$$

on F the following assumptions hold

1. $F \in C^1(\mathbb{R}^2, \mathbb{R})$ and $F(0) = 0$;
2. $|DF(u)| \leq c(|u|^{p-1} + |u|^{q-1})$ with $2 < p \leq q < 2^*$;
3. $F(u_1, u_2) = -\beta |u_1 u_2|^\gamma + G(u)$, $\beta > 0$, $2 < 2\gamma < p$;
4. $G \geq 0, \ G(u_1, u_2) = G(|u_1|, |u_2|)$;
5. $V(u) := m_1^2 u_1^2/2 + m_2^2 u_2^2/2 + F(u) \geq 0$;
6. G is well-behaved with respect to the symmetric rearrangement

$$\int_{\mathbb{R}^N} G(u_1^*, u_2^*) \leq \int_{\mathbb{R}^N} G(u_1, u_2).$$

We are looking for standing-wave pairs solutions to (CNLKG).

$$v_j(t,x) = u_j(x)e^{-i\omega_j t}, \quad 1 \le j \le 2$$

In particular, a solution $(u,\omega) \in H^1(\mathbb{R}^N, \mathbb{R}^2) \times \mathbb{R}^2$

$$\begin{aligned} -\Delta u_1 + (m_1^2 - \omega_1^2)u_1 + \partial_{v_1} F(u) = 0 \\ -\Delta u_2 + (m_2^2 - \omega_2^2)u_2 + \partial_{v_2} F(u) = 0 \\ u_j > 0 \end{aligned}$$

Furthermore, (u, ω) has the following variational characterisation

$$E(u, \omega) = \inf_{M_c} E =: I(c)$$

for some $c \in \mathbb{R}^2$ with $c_j > 0$.

$$E\colon H^1(\mathbb{R}^N, \mathbb{R}^2) \times \mathbb{R}^2 \to \mathbb{R}$$
$$(v, \alpha) \mapsto \frac{1}{2} \sum_{j=1}^{2} \|Dv_j\|^2 + m_j^2 \|v_j\|^2 + \int_{\mathbb{R}^N} F(v)$$

$$C_j\colon H^1(\mathbb{R}^N, \mathbb{R}^2) \times \mathbb{R}^2 \to \mathbb{R}$$
$$(v, \alpha) \mapsto \alpha_j \|v_j\|^2, \quad 1 \leq j \leq 2$$

$$M_c = \{(v, \alpha) \mid C_j(v, \alpha) = c_j\}.$$

Given

$$(\phi, \phi_t) \in H^1(\mathbb{R}^N, \mathbb{C}^2) \oplus L^2(\mathbb{R}^N, \mathbb{C}^2) =: X$$

there exists $T > 0$ and a unique

$$v \in C_t H^1_x(0, T; \mathbb{R}^N) \cap C^1_t L^2_x(0, T; \mathbb{R}^N)$$

such that v solves (CNLKG) and

$$v(0) = \phi, \quad v'(0) = \phi_t.$$

In the scalar case the problem was addressed by J. Ginibre and G. Velo (Math. Zeit., 1985).

From the assumption

$$F(u_1, u_2) = F(|u_1|, |u_2|)$$

we have conserved quantities

$$\begin{aligned} \mathbf{E}\colon X &\to \mathbb{R}, \qquad \text{(Energy)} \\ (\phi, \phi_t) &\mapsto \frac{1}{2}\int_{\mathbb{R}^N} |\phi_t|^2 + |D\phi|^2 \\ &\qquad + m_1^2|\phi_1|^2 + m_2^2|\phi_2|^2 + 2F(\phi) \end{aligned}$$

$$\begin{aligned} \mathbf{C}_j\colon X &\to \mathbb{R}, \quad 1 \le j \le 2 \\ (\phi, \phi_t) &\mapsto -\mathrm{Im}\int_{\mathbb{R}^N} \phi_t^j \overline{\phi}_j. \end{aligned} \qquad \text{(Charges)}$$

If $(\phi, \phi_t) = (u_1, u_2, -i\omega_1 u_1, -i\omega_2 u_2)$, then $\mathbf{E}$ and $\mathbf{C}_j$ correspond to $E(u, \omega)$ and $C_j(u, \omega)$.

Orbital Stability

A subset $S \subset X$ is said stable if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, if

$$\mathrm{dist}(p, S) < \delta$$

then the local solution v of (CNLKG) with initial datum p is defined on $[0, +\infty)$ and

$$\mathrm{dist}((v(t), v'(t)), S) < \varepsilon, \quad t \geq 0.$$

An initial datum $q \in X$ is said orbitally stable if there exists a closed, finite-dimensional manifold S such that

1. S is positively invariant;
2. S is stable.

We say that the standing-waves pair solution to (CNLKG)

$$(u_1, u_2, -i\omega_1 u_1, -i\omega_2 u_2)$$

is orbitally stable if

$$\Gamma(u,\omega) = \left\{ \begin{array}{c} \Big(\lambda_1 u_1(\cdot + y), \lambda_2 u_2(\cdot + y), \\ (-i\lambda_1\omega_1 u_1(\cdot + y)), -i\lambda_2\omega_2 u_2(\cdot + y))\Big) \\ \\ (\lambda, y) \in S^1 \times S^1 \times \mathbb{R}^N. \end{array} \right.$$

is stable. Given $c \in \mathbb{R}^2$, we define

$$\Gamma_c := \bigcup_{\substack{E(u,\omega)=I(c) \\ (u,\omega)\in M_c}} \Gamma(u,\omega)$$

called *ground state* (V. Benci, J. Bellazzini *et al.*, Adv. Nonlinear Stud., 2010).

J. Shatah 1983 (Comm., Math, Physics, 1983)
NLKG, least energy solutions; orbital stability in $H_r^1(\mathbb{R}^N)$, $N \geq 3$ with $F(u) = -|u|^{p-1}u$ and $p < 1 + 4/N$. Stable for $\omega \in (\omega^*, 1)$;

J. Bellazzini, V. Benci, C. Bonanno, M. Micheletti (Adv. Nonlinear Stud., 2010)
NLKG, $E(u, \omega) = \inf_{M_c} E$; stability of the ground state and $\Gamma(u, \omega)$ (under a non-degeneracy assumption); scalar case and $N \geq 3$;

J. Zhang, Z. Gan, B. Guo 2010 (Acta Math. Appl. Sin. Engl. Ser., 2010)
CNLKG; analogous results to the Shatah's work, $F(u_1, u_2) = -|u_1|^{p+1}|u_2|^{q+1}$, $\omega_1 = \omega_2 \in (\omega^*, 1)$.

Theorem

1. *For every c such that* $c_j > 0$ *the ground state is stable;*
2. *for every* (u, ω) *such that* $E(u, \omega) = I(c)$ *and there exists* $r_0 > 0$ *such that*

$$\Gamma(v, \alpha) \neq \Gamma(u, \omega) \Rightarrow B(\Gamma(u, \omega), r_0) \cap \Gamma(v, \alpha) = \emptyset$$

   $\Gamma(u, \omega)$ *is stable (i.e. the corresponding standing-wave is orbitally stable).*

The proof is carried out by contradiction: let $(\Phi_n) \subset X$ $\delta > 0$ and $(t_n) \subset \mathbb{R}$ be such that

$$\operatorname{dist}(\Phi_n, \Gamma_c) \to 0, \quad \operatorname{dist}((v_n(t_n), v_n'(t_n)), \Gamma_c) \geq \delta.$$

We know that

$$\mathbf{E}((v_n(t_n), v_n'(t_n)) \to I(c), \quad \mathbf{C}_j((v_n(t_n), v_n'(t_n)) \to c_j.$$

Theorem

Let $(\Psi_n) \subset X$ be a sequence such that

$$\mathbf{E}(\Psi_n) \to I(c), \quad \mathbf{C}_j(\Psi_n) \to c_j.$$

Then

$$\operatorname{dist}(\Psi_n, \Gamma_c) \to 0.$$

See also J. Bellazzini, V. Benci *et al.* (Adv. Nonlinear Stud., 2010).

Theorem

Given a minimising sequence (u_n, ω_n) *of* E *on* M_c*, then*

$$u_{n_k} = u(\cdot + y_k) + o(1) \text{ in } H^1(\mathbb{R}^N, \mathbb{R}^2), \quad \omega_{n_k} \to \omega$$

for some (u, ω) *such that* $E(u, \omega) = I(c)$ *and* $(y_k) \subset \mathbb{R}^N$*.*

In turn, the preceding Theorem follows from:

1. the Lemma I.1 of P.L. Lions (Ann. Inst. H. Poincaré Anal. Non Linéaire 1, 1984, no. 2). The term

$$-\beta|u_1 u_2|^\gamma$$

rules out the vanishing case;

2. given c, c', there exists $\varepsilon > 0$ and $d = d(\sigma, \tau) \in (0,1)$ given non-negative radially symmetric

$$((u,\omega),(u',\omega')) \in M_c \times M_{c'}$$

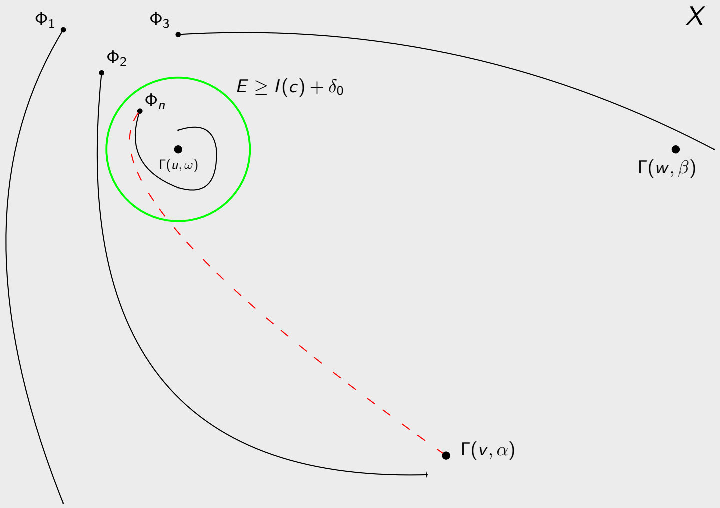
with compact and disjoint support

$$E(u,\omega) \le I(c) + \varepsilon, \quad E(u',\omega') \le I(c') + \varepsilon.$$

$$\|D(u+u')^*\|^2 \le c(\|Du\|^2 + \|Du'\|^2)$$

3. $E(u_n - u, \omega) = E(u_n, \omega) - E(u, \omega) + o(1)$ if $u_n \rightharpoonup u$.

The stability of $\Gamma(u,\omega)$



The non-degeneracy condition

It is interesting to know whether the non-degeneracy condition can be dropped from our assumption.

1. it is possible to drop the non-degeneracy requirement to obtain the orbital stability of $\Gamma(u, \omega)$;
2. or, are there solutions to a NLKG which connect points arbitrarily close to $\Gamma(u, \omega)$ to points close to $\Gamma(v, \alpha)$?

Acknowledgements

- Professor Vieri Benci and Professor Jaeyoung Byeon for their helpful suggestion;
- this work was supported by the Priority Research Centers Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (Grant #2009-0094068);

- thank you for your attention.