Minimal stable subsets of the ground state: the non-linear Schrödinger equation

Daniele Garrisi* [1] Vladimir Georgiev [2]

[1] Inha University, College of Mathematics Education

[2] Università degli Studi di Pisa, Dipartimento di Matematica "Leonida Tonelli"

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We consider the non-linear Schrödinger equation

$$\text{(NLS)} \qquad (i\partial_t + \Delta)\phi + g(\phi) = 0$$

where

$$\phi \colon \mathbb{R}_t \times \mathbb{R}_x \to \mathbb{C}, \; g \colon \mathbb{C} \to \mathbb{C}$$

such that

$$g(zu) = zg(u)$$

for every pair (z, u) in $\mathbb{C}^2$ such that $|z| = 1$ $(z \in S^1)$.

Let $G \colon \mathbb{C} \to \mathbb{R}$ be such that for every $s \geq 0$

$$G'(s) = g(s), \quad G(0) = 0.$$

We assume that (NLS) is globally well-posed in $H^1(\mathbb{R};\mathbb{C})$:

That is, given Φ in $H^1(\mathbb{R};\mathbb{C})$, there exists only one solution

$$\phi\colon [0,+\infty)\times\mathbb{R}_x \to \mathbb{C}$$

to (NLS) such that

$$\phi(0,x)=\Phi(x), \quad \phi(t,\cdot)\in H^1(\mathbb{R};\mathbb{C}).$$

The notation

$$U_t(\Phi)=\phi(t,\cdot), \quad U_t\colon H^1(\mathbb{R};\mathbb{C})\to H^1(\mathbb{R};\mathbb{C})$$

for every $t\geq 0$ is useful.

Definition (Stable subsets of $H^1(\mathbb{R};\mathbb{C})$)

A subset $S \subseteq H^1$ is stable if for $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\operatorname{dist}(\Phi, S) < \delta \Rightarrow \operatorname{dist}(U_t(\Phi), S) < \varepsilon$$

for every Φ in H^1 and for every $t \geq 0$.

$$\operatorname{dist}(\Phi, S) := \inf_{\Psi \in S} \|\Phi - \Psi\|_{H^1(\mathbb{R};\mathbb{C})}.$$

The distance induced by a scalar product

$$\langle \Phi, \Psi \rangle_{H^1_{\mathbb{C}}(\mathbb{R})} := \operatorname{Re} \int_{-\infty}^{+\infty} \Phi(x)\overline{\Psi(x)}dx.$$

Conserved quantities and symmetries

Given Φ, we define the energy

$$E(\Phi) := \int_{-\infty}^{+\infty} |\Phi'(x)|^2 dx + \int_{-\infty}^{+\infty} G(\Phi(x)) dx$$

and the charge

$$C(\Phi) := \mathrm{Re} \int_{-\infty}^{+\infty} \Phi(x)\overline{\Phi}(x) dx = \|\Phi\|_{L^2}^2.$$

The functions

$$e(t) := E(U_t(\Phi)), \quad c(t) := C(U_t(\Phi)).$$

are constant. Moreover, given $|z| = 1$ and $y \in \mathbb{R}$

$$E(z\Phi(\cdot + y)) = E(\Phi), \quad C(z\Phi(\cdot + y)) = C(\Phi).$$

Solitary waves

A solitary wave is a solution ϕ to (NLS) such that

$$\phi_v(t,x) = e^{i(\omega-|v|^2)t+iv\cdot x}u(x-2tv),$$

where

$$v \in \mathbb{R}, \quad \omega \in \mathbb{R}, \quad u \in H^1(\mathbb{R};\mathbb{R}).$$

When $v = 0$, ϕ is also called standing-wave:

$$\phi(t,x) = e^{i\omega t}u(x).$$

If ϕ_v is a solution to (NLS), then

$$\text{(E)} \qquad u'' - g(u) - \omega u = 0.$$

The profile u is obtained as minimum of E

$$E(u) := \frac{1}{2}\int_{-\infty}^{+\infty} |u'(x)|^2 dx + \int_{-\infty}^{+\infty} G(u(x))dx$$

on the constraint

$$S(\lambda) = \{u \in H^1(\mathbb{R};\mathbb{C}) \mid C(u) = \lambda\}.$$

The second order ODE is

$$\nabla E(u) = \omega \nabla C(u).$$

The sets G_λ and $G_\lambda(u)$

Definition (The ground state)

Given $\lambda > 0$, we define

$$G_\lambda := \{\Phi \mid E(\Phi) = \min_{S(\lambda)} E\}$$

and

$$G_\lambda(u) := \{zu(\cdot + y) \mid |z| = 1, y \in \mathbb{R}\}$$

for every $u \in G_\lambda$.

(1) Existence of solitary waves ($G_\lambda \neq \emptyset$)

(2) stability of $G_\lambda \subseteq H^1(\mathbb{R};\mathbb{C})$ (ground state)

(3) stability of $G_\lambda(u) \subseteq H^1(\mathbb{R};\mathbb{C})$

(1) and (2) follow from the following fact:

$$E(\Phi_n) \to \inf(E)$$

there exists a sequence (y_n) and Φ such that

$$\Phi_n(\cdot + y_n) \to \Phi \text{ in } H^1$$

from the Concentration-Compactness Lemma (Lions, 1984).

If

$$|g(s)| \le c(|s|^p + |s|^q), \quad 2 < p \le q < 6$$

and

$$\exists s_0 \text{ such that } G(s_0) < 0$$

then G_λ is stable.

(Benci, *et al.*, Advanced Nonlinear Studies, 2007).

(3) Stability of $G_\lambda(u)$

Pure power case: $g(s) = -|s|^{p-2}u$, $p > 2$

Cazenave and Lions (Comm. Math. Phys., 1982).

For every u there holds

$$G_\lambda(u) = G_\lambda.$$

In fact,

$$u_\omega(x) = z\omega^{1/(p-1)}u_1(\omega^{1/2}(x+y))$$

where u_1 is the unique positive solution in H^1 to

$$u_1'' - u_1 - g(u_1) = 0$$

such that $u_1(x) = u_1(-x)$, and $(z, y) \in S^1 \times \mathbb{R}$.

So, $G_\lambda(u)$ is stable because G_λ is stable.

If g is a general non-linearity, $G_\lambda(u)$ is stable provided

$$\text{(B3)} \qquad \int_{-\infty}^{+\infty} \left(\frac{g(u(x))}{u(x)} \cdot (1 - u'(x)^2) + u'(x)^2 g'(u(x)) \right) dx \neq 0.$$

M. Weinstein (Comm. Math. Phys., 1986).

So far, we could remove this condition.

Theorem (G., Georgiev)

There are finitely many $G_\lambda(u)$. Each of them is stable.

The Hessian of E (restricted on $S(\lambda)$) is positively defined (oscillation theory).

Minima are isolated.

We do not know whether $G_\lambda(u) = G_\lambda$.

$G_\lambda(u)$ contains the orbit of the standing-wave

$$\phi(t,x) = e^{i\omega t}u(x)$$

as

$$G^*_\lambda(u) := \{zu \mid |z| = 1\} \subseteq G_\lambda(u) \subseteq G_\lambda.$$

However, $G^*_\lambda(u)$ is not stable (Cazenave and Lions, CPM, 1982).

A set $S \subseteq H^1(\mathbb{R};\mathbb{C})$ is invariant if

$$U_t(S) \subseteq S$$

for every $t \geq 0$.

Corollary (Minimality of $G_\lambda(u)$, G., Georgiev)

Given a closed, stable and invariant set such that

$$G^*_\lambda(u) \subseteq S \subseteq G_\lambda$$

there holds $G_\lambda(u) \subseteq S$.