Uniqueness and non-degeneracy of Q-balls in dimension one

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A standing-wave

$$\phi(t,x) = e^{i\omega t} R(x), \quad \omega \in \mathbb{R}$$

is a solution to the non-linear Schrödinger equation

$$\text{(NLS)} \qquad \big(i\partial_t \phi + \Delta_x \phi\big)(t,x) + g(\phi(t,x)) = 0$$

where R minimizes the energy on a mass constraint.

A Q-ball is a standing-wave such that R is in $H^1_{r,+}(\mathbb{R}^n;\mathbb{R})$ and

(1) $R(x) > 0$ for every $x \in \mathbb{R}$
(2) $R(x) = R(x')$ if $|x| = |x'|$
(3) $R, |\nabla R| \in L^2$

The expression Q-ball was introduced by Rosen (J. Math. Phys., 1968).

Here we will refer to the profile R with the same expression.

We define the energy functional

$$E(u) := \frac{1}{2}\int_{-\infty}^{+\infty} |\nabla u(x)|^2 dx + \int_{-\infty}^{+\infty} G(u(x))dx$$

on the mass constraint

$$M(u) := \int_{-\infty}^{+\infty} |u(x)|^2 dx, \quad S(\lambda) := \{u \mid M(u) = \lambda\}.$$

Both E and M are defined on $X := H^1_{r,+}(\mathbb{R}^n; \mathbb{C})$.

We define $I(\lambda) := \inf_{S(\lambda)} E$ and

$$\mathcal{G}_\lambda := \{u \in X \cap S(\lambda) \mid E(u) = I(\lambda)\}$$

which is the set of minima of E over S.

Q-balls play a role in the stability of standing-waves.

(1) if every $R \in \mathcal{G}_\lambda$ is a non-degenerate critical point of E over S
(2) if $\mathcal{G}_\lambda$ consists of a single point (uniqueness)

then all the standing-waves are stable.

(2) is equivalent to

(2E) given $R_1, R_2 \in \mathcal{G}_\lambda$ and $\omega_1, \omega_2 \in \mathbb{R}$

$$\Delta R_1(x) - G'(R_1(x)) - \omega_1 R_1(x) = 0$$
$$\Delta R_2(x) - G'(R_2(x)) - \omega_2 R_2(x) = 0$$

implies $R_1 = R_2$ and $\omega_1 = \omega_2$.

Hereafter, we restrict to the dimension $n = 1$.

The non-degeneracy

(1) is equivalent to

(1E) For every $v \in H^1_r$ and $\beta \in \mathbb{R}$

$$L(v) = v'' - G''(R)v - \omega v = \beta v \Rightarrow \beta = 0 \text{ and } v = 0.$$

M. Weinstein, Comm. Math. Phys., 1985, pure power case

If $G(s) = -a|s|^p$ with $2 < p < 6$ and $a > 0$, then R is non-degenerate.

M. Weinstein, Comm. Math. Phys., 1986

R is non-degenerate, provided

$$\text{(B3)} \quad \int_{-\infty}^{+\infty} \left(\frac{G'(R(x))}{R(x)} \cdot (1 - R'(x)^2) + R'(x)^2 G''(R(x)) \right) dx \neq 0.$$

The second result applies to general non-linearities.

Our goal: a result based on assumptions on G (e.g. pure powers).

G. and Georgiev

If for every $R \in \mathcal{G}_\lambda$ there holds

$$12G(s) - 7sG'(s) + s^2G''(s) \geq 0$$

for every $s \in \mathrm{Im}(R)$, then every R is non-degenerate.

A one-parameter family R_ω is build

$$R''_\omega - G'(R_\omega) - \omega R_\omega = 0, \quad \lambda(\omega) := \|R_\omega\|^2_{L^2}.$$

$$\lambda'(\omega) = \langle L(\partial_\omega R_\omega), \partial_\omega R_\omega \rangle_{L^2} \geq 0.$$

$\lambda'(\omega) > 0$ gives the non-degeneracy. If $\lambda'(\omega) = 0$, then

$$12G(s) - 7sG'(s) + s^2G''(s) = 0 \Rightarrow G(s) = cs^2 + ds^6$$

which is the critical case where there are no Q-balls.

The proof is based on the $d(\omega)$ function of W. Strauss *et al.*, CMP, 1985.

Uniqueness

If G is a pure-power, and

$$R'' - G'(R) - \omega R = 0, \quad R \in \mathcal{G}_\lambda$$

the rescaling property $G(ts) = t^p G(s)$ for $t > 0$ implies

$$R(x) = \omega^{1/(p-1)} R_1(\omega^{1/2} x)), \quad \omega = (\lambda \|R_1\|_2^{-2})^{\frac{2(p-1)}{(5-p)}}$$

and R_1 is the solution in $H^1_{r,+}$ to

$$R_1''(x) - G'(R_1(x)) - R_1(x) = 0$$

which is unique (H. Berestycki and P.L. Lions, ARMA, 1983).

General non-linearities

Non-degeneracy implies that $\mathcal{G}_\lambda$ is a finite set.

Obstructions to the uniqueness: two one-parameter families

$$R\colon (\omega_1, \omega_2) \to H^1, \quad R_*\colon (\omega_1^*, \omega_2^*) \to H^1$$

such that $\omega_2 \leq \omega_1^*$ there are ω, ω_* satisfying

$$\|R_\omega\|_2^2 = \|R_{\omega_*}\|_2^2, \quad E(R_\omega) = E(R_{\omega_*})$$

The multiplicity of these intervals is related to the critical points of

$$V(s) := -\frac{2G(s)}{s^2}.$$

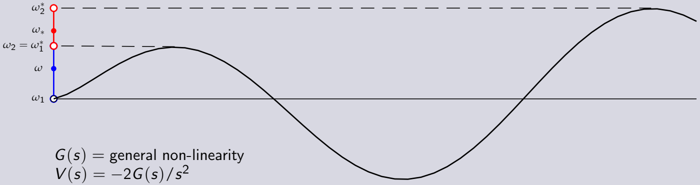
If (R, ω) satisfies

$$R'' - G'(R) - \omega R = 0$$

then there exists s_* such that

$$\omega = V(s_*), \quad V'(s_*) > 0.$$

Graph of V

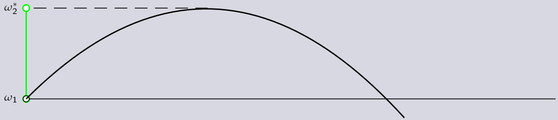


$G(s) =$ general non-linearity
$V(s) = -2G(s)/s^2$
$\#\mathcal{G}_\lambda \leq 2$

G. and Georgiev, $G(s) = -a|s|^p + b|s|^q$, $2 < p < 6$ and $p < q$

G satisfies the Euler differential inequality. There is only one, non-degenerate Q-ball.

Graph of V



$G(s) = -a|s|^p + b|s|^q$
$V(s) = 2a|s|^p - 2b|s|^q$
$\#\mathcal{G}_\lambda = 1$

G. and Georgiev, $G(s) = -a|s|^p + b|s|^q$, $2 < p < 6$ and $p < q$

G satisfies the Euler differential inequality. There is only one, non-degenerate Q-ball.