EXERCISES OF WEEK FIVE

Exercise 1. Given three lines $\ell_1 := \ell(P,v), \ell_2 := \ell(Q,w)$ and $\ell_3 := \ell(R,z)$ such that

$$\ell_i \neq \ell_j \text{ if } i \neq j$$

find necessary and sufficient conditions in terms of $P, Q, R \in \mathbb{R}^3$ and $v, w, z \in E^3$ in order to have

$$\ell_1 \cap \ell_2 \cap \ell_3 \neq \emptyset.$$

Solution. There is intersection between the three lines if and only if

$$\ell_1 \cap \ell_2 \neq \emptyset$$

and the intersection point R belongs to ℓ_3 as well.
Since $\ell_1 \neq \ell_2$, the intersection is non-empty if and only if

$$\overrightarrow{PQ} \cdot (v \times w) = 0, \quad v \times w \neq 0.$$

In this case the intersection consists of a single point which is given by the formula

$$T = Q - \frac{(v \times \overrightarrow{PQ}) \cdot (v \times w)}{\|v \times w\|^2} w. \tag{1}$$

Now, we need $R \in \ell_3$, which implies

$$T = R + tz$$

for some $t \in \mathbb{R}$. Since $z \neq 0$, this is equivalent to

$$\overrightarrow{RT} \times z = 0. \tag{2}$$

From (1)

$$\overrightarrow{RT} = \overrightarrow{QR} - \frac{(v \times \overrightarrow{PQ}) \cdot (v \times w)}{\|v \times w\|^2} w.$$

Then, from (2) a necessary and sufficient condition to have intersection of the three lines is

$$v \times w \neq 0$$

$$\overrightarrow{PQ} \cdot v \times w = 0$$

$$\left(\overrightarrow{QR} - \frac{(v \times \overrightarrow{PQ}) \cdot (v \times w)}{\|v \times w\|^2} w \right) \times z = 0$$

□

Exercise 2. We define the two-variable function

$$g(x,y) = \begin{cases} \frac{x^2 y}{x^2+y^2} & \text{if } x^2+y^2 \neq 0 \\ 0 & x = y = 0 \end{cases}$$

State whether[1]

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[1] The inequality $2xy \leq x^2 + y^2$ is useful in this exercise

1. g is bounded on $B((0,0),1)$
2. g is continuous at the point $O(0,0)$
3. the partial derivatives $\partial_x g(O)$ and $\partial_y g(O)$ exist
4. the partial derivatives $\partial_x g$ and $\partial_y g$ are bounded on $B((0,0),1)$
5. g is differentiable at $O(0,0)$
6. g is smooth on $B((0,0),1)$

Solution. 1. g is bounded. If $(x,y) \in B(O,1)$ and $x^2+y^2 \neq 0$, we have

$$\left|\frac{x^2y}{x^2+y^2}\right| = |x| \cdot \left|\frac{xy}{x^2+y^2}\right| \le \frac{|x|}{2} \cdot \frac{x^2+y^2}{x^2+y^2} = \frac{|x|}{2} \le \frac{1}{2}. \tag{3}$$

because $|x| \le \sqrt{x^2+y^2} \le 1$. If $x^2+y^2=0$, then $x=y=0$ and $g(0,0)=0$. Then g is bounded on the unit ball and

$$|g| \le 1/2$$

2. g is continuous at O. In fact, from (3)

$$|g(x,y)-g(0,0)| = |g(x,y)| \le \frac{|x|}{2} \le \frac{\|(x,y)\|}{2}.$$

Then

$$\lim_{(x,y)\to(0,0)} g(x,y)-g(0,0) = 0$$

3. the partial derivatives at O exist. Firstly, we evaluate $\partial_x g(O)$. We have

$$\lim_{t\to 0} \frac{g(t,0)-g(0,0)}{t} = 0 = \partial_x g(O). \tag{4}$$

Similarly,

$$\lim_{t\to 0} \frac{g(0,t)-g(0,0)}{t} = 0 = \partial_y g(O) \tag{5}$$

4. the partial derivative $\partial_x g$ at a point $(x,y) \neq O$ is

$$\partial_x g = \frac{2xy(x^2+y^2) - x^2y \cdot 2x}{(x^2+y^2)^2} = \frac{2xy^3}{(x^2+y^2)^2}.$$

Then

$$\partial_x g(x,y) = \begin{cases} \frac{2xy^3}{(x^2+y^2)^2} & \text{if } x^2+y^2 \neq 0 \\ 0 & \text{if } x=y=0. \end{cases}$$

As for the partial derivative $\partial_y g$, if $(x,y) \neq O$, we have

$$\partial_y g(x,y) = \frac{x^2(x^2+y^2) - x^2y \cdot 2y}{(x^2+y^2)^2} = \frac{x^4 - x^2y^2}{(x^2+y^2)^2}.$$

Then

$$\partial_y g(x,y) = \begin{cases} \frac{x^4-x^2y^2}{(x^2+y^2)^2} & \text{if } x^2+y^2 \neq 0 \\ 0 & \text{if } x=y=0. \end{cases}$$

We have

$$|\partial_x g(x,y)| = \left|\frac{2xy^3}{(x^2+y^2)^2}\right| = \frac{2|xy|}{x^2+y^2}\cdot\frac{y^2}{x^2+y^2}$$
$$\leq \frac{x^2+y^2}{x^2+y^2}\cdot\frac{x^2+y^2}{x^2+y^2} = 1$$

Then, if $x \neq 0$

$$|\partial_x g(x,y)| \leq 1.$$

If $x = 0$ and $y \neq 0$, then $\partial_x g(x,y) = 0$. If $x = y = 0$, from (3) $\partial_x g = 0$. In conclusion

$$|\partial_x g| \leq 1 \text{ on } B(O,1).$$

As for $\partial_y g$, we have

$$|\partial_y g(x,y)| = \left|\frac{x^4 - x^2y^2}{(x^2+y^2)^2}\right| \leq \frac{x^4 + x^2y^2}{x^4 + 2x^2y^2 + y^4} \leq 1$$

if $x^2 + y^2 = 0$. If $x = y = 0$, $\partial_y g = 0$. Then

$$|\partial_y g| \leq 1 \text{ on } B(O,1)$$

5. g is not differentiable at O. In fact, we can evaluate the directional derivatives of g at a vector $v \neq 0$

$$\partial_v g(O) = \lim_{t\to 0}\frac{g(tv_1, tv_2)}{t} = \frac{t^2v_1^2\cdot tv_2}{t^3(v_1^2+v_2^2)} = \frac{v_1^2v_2}{v_1^2+v_2^2}.$$

If g is differentiable at O, then we had

$$\partial_v g(O) = v_1\partial_x g(O) + v_2\partial_y g(O).$$

However, the partial derivatives of g at the origin are zero. So the equality above fails if

$$\frac{v_1^2v_2}{v_1^2+v_2^2} \neq 0$$

for instance, if $v_1v_2 \neq 0$

6. the function g is not smooth on $B(0,1)$, because $\partial_x g$ is not continuous at O. In fact, two different sequences

$$P_n = (1/n, 0)$$

and

$$Q_n = (1/n, 1/n)$$

give two different limits

$$\partial_x g(P_n) = 0 \Rightarrow \lim_{n\to\infty}\partial_x g(P_n) = 0$$

while

$$\partial_x g(Q_n) = 1/2 \Rightarrow \lim_{n\to\infty}\partial_x g(Q_n) = 1/2.$$

$\square$