SOLUTIONS OF EXERCISES OF WEEK TEN

Exercise 1. Given the function

$$g\colon \mathbb{R} \to (-\pi/2, \pi/2), \quad g(s) = \arctan(s)$$

show that

$$g(1/s) = \frac{\pi}{2} - g(s)$$

for every $s > 0$ and

$$g(1/s) = -\frac{\pi}{2} - g(s)$$

for every $s < 0$.

Solution. When $s > 0$ we have two functions defined on $(0, +\infty)$

$$h_1^+(s) = g(1/s), \quad h_2^+(s) = \frac{\pi}{2} - g(s)$$

We have

$$h_1^{+\prime}(s) = g'(1/s) \cdot -\frac{1}{s^2} = \frac{1}{1+1/s^2} \cdot -\frac{1}{s^2} = -\frac{1}{1+s^2}$$
$$h_2^{+\prime}(s) = -g'(s) = -\frac{1}{1+s^2}.$$

Then

$$h_1^{+\prime} \equiv h_2^{+\prime}$$

on $(0, +\infty)$. Then, there exists a constant $c > 0$ such that

$$h_2^+(s) - h_1^+(s) = c.$$

Taking the limit as $s \to +\infty$, we obtain

$$c = \lim_{s\to+\infty}(h_2^+(s) - h_1^+(s)) = \lim_{s\to+\infty} h_2^+(s) - \lim_{s\to+\infty} h_1^+(s) = 0 - 0 = 0.$$

In order to obtain the second equality, we define

$$h_1^-(s) = g(1/s), \quad h_2^-(s) = -\frac{\pi}{2} - g(s).$$

Then $h_1^{-\prime} \equiv h_2^{-\prime}$ and there exists a constand d such that

$$h_2^-(s) - h_1^-(s) = c.$$

Taking the limit as $s \to -\infty$ we obtain

$$d = \lim_{s\to-\infty}(h_2^-(s) - h_1^-(s)) = \lim_{s\to-\infty} h_2^-(s) - \lim_{s\to-\infty} h_1^-(s) = 0 - 0 = 0.$$

□

Exercise 2. Find the potential of the vector field

$$\mathbf{X} = \frac{1}{2\pi}\left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$$

on the following regions:

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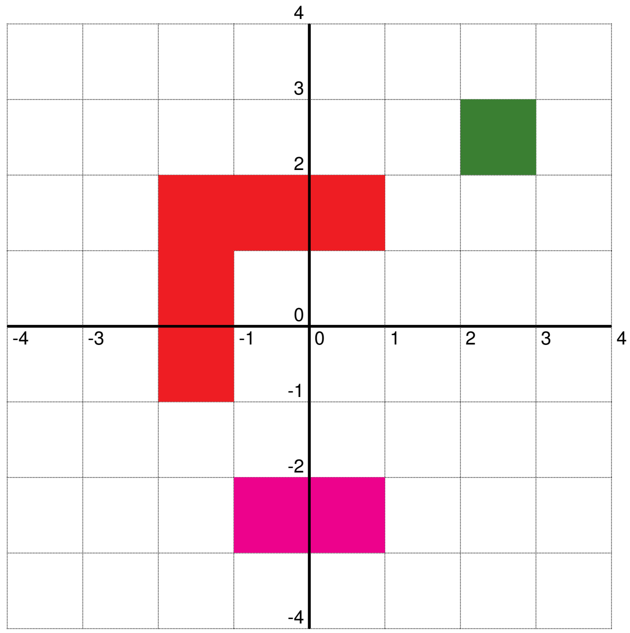

$\Omega_1 := \{(x,y) \mid 2 < x < 3, 2 < y < 3\}$
$\Omega_2 := \{(x,y) \mid -1 < x < 1, -3 < y < -2\}$
$\Omega_3 := \{(x,y) \mid -2 < x < 1, 1 < y < 2\} \cup \{(x,y) \mid -2 < x < 2, -3 < y < -2\}$

Solution. Ω_1. The function

$$g_1(x,y) = \arctan\frac{y}{x}$$

is smooth on Ω_1 and $\nabla g_1 = \boldsymbol{X}$.

Ω_2. In this the domain $x = 0$. Then, it is convenient to use a different representation of the arctan. In Ω_2, $y < 0$. Then, from the first exercise

$$\arctan\frac{y}{x} = -\frac{\pi}{2} - \arctan\frac{x}{y}$$

The function

$$g_2(x,y) = -\frac{\pi}{2} - \arctan\frac{x}{y}$$

it is defined on Ω_2 and $\nabla g_2 = \boldsymbol{X}$.

Ω_3. The open set can be divided in two different regions:

$$\Omega_3 \cap \{y < 0\}, \quad \Omega_3 \cap \{y \geq 0\}.$$

We define

$$g_3(x,y) = \begin{cases} \arctan\frac{x}{y} & \text{if } y < 0 \\ \frac{\pi}{2} & \text{if } y = 0 \\ \arctan\frac{x}{y} + \pi & \text{if } y > 0 \end{cases}$$

We check that the correction $+\pi$ makes the function g_3 continuous

$$\lim_{y\to 0, y<0} g_3(x,y) = \frac{\pi}{2}$$

$$\lim_{y\to 0, y>0} g_3(x,y) = -\frac{\pi}{2} + \pi = \pi/2.$$

Cleary, if $y \neq 0$

$$\nabla g_3 = \boldsymbol{X}.$$

We show that the equality above holds when $y = 0$ as well. We have

$$\frac{g_3(x+s,0) - g_3(x,0)}{s} = \frac{\pi/2 - \pi/2}{s} = 0 \Rightarrow \partial_x g(x,0) = 0;$$

as for the partial derivative with respect to y, suppose that $s > 0$. Then

$$\lim_{s \to 0} \frac{g_3(x,s) - g_3(x,0)}{s} = \lim_{s \to 0} \frac{\arctan \frac{x}{s} + \pi - \pi/2}{s} = -\frac{1}{x^2}.$$

If $s < 0$,

$$\lim_{s \to 0} \frac{g_3(x,s) - g_3(x,0)}{s} = \lim_{s \to 0} \frac{\arctan \frac{x}{s} - \pi/2}{s} = -\frac{1}{x^2}.$$

Since the limit is the same, the partial derivative exists and

$$\partial_y g_3(x,0) = -\frac{1}{x^2}.$$

Then

$$\nabla g_3(x,0) = \left(0, -\frac{1}{x^2}\right) = \boldsymbol{X}(x,0).$$

□

Exercise 3. An ellipse of axes a and b can be parametrized with the curve

$$\alpha \colon [0,1] \to \mathbb{R}^2, \quad \alpha(t) = (a\cos 2\pi t, b \sin 2\pi t)$$

Using the Green's theorem, find the area of the ellipse.

Solution. The area of the ellipse is

$$\begin{aligned}\oint_\alpha x dy &= \int_0^1 a\cos 2\pi t \cdot 2\pi b \cos 2\pi t dt \\ &= 2\pi ab \int_0^1 \cos^2 2\pi t dt = 2\pi ab \cdot \frac{1}{2} \int_0^1 (1 - \cos 4\pi t) dt = \pi ab\end{aligned}$$

□