SOLUTIONS OF THE EXERCISES OF WEEK TWELVE

Exercise 1. Given open sets $\Omega, \Omega' \subseteq \mathbb{R}^2$ and C^1 function

$$\psi\colon \Omega \to \mathbb{R}^3, \quad \varphi\colon \Omega' \to \Omega$$

we can define the composition

$$g(u,v) := \psi(\varphi(u,v)).$$

Using the chain rule (check Theorem 4.11, page 166 of the book of M. Corral), show that

$$\partial_u g(u,v) \times \partial_v g(u,v) = \big(\partial_u \varphi(u,v) \times \partial_u \varphi(u,v)\big)(\partial_x \psi(x,y) \times \partial_y \psi(x,y)).$$

Solution. By the chain rule,

$$\partial_u g = \partial_x \psi \partial_u \varphi_1 + \partial_y \psi \partial_u \varphi_2, \quad \partial_v g = \partial_x \psi \partial_v \varphi_1 + \partial_y \psi \partial_v \varphi_2.$$

Then

$$\begin{aligned}
\partial_u g \times \partial_v g &= (\partial_x \psi \partial_u \varphi_1 + \partial_y \psi \partial_u \varphi_1) \times (\partial_x \psi \partial_v \varphi_1 + \partial_y \psi \partial_v \varphi_1) \\
&= \partial_x \psi \partial_u \varphi_1 \times \partial_x \psi \partial_v \varphi_1 + \partial_x \psi \partial_u \varphi_1 \times \partial_y \psi \partial_v \varphi_2 \\
&+ \partial_y \psi \partial_u \varphi_2 \times \partial_x \psi \partial_v \varphi_1 + \partial_y \psi \partial_u \varphi_2 \times \partial_y \psi \partial_v \varphi_2 \\
&= 0 + \partial_x \psi \partial_u \varphi_1 \times \partial_y \psi \partial_v \varphi_2 + \partial_y \psi \partial_u \varphi_2 \times \partial_x \psi \partial_v \varphi_1 + 0 \\
&= (\partial_u \varphi_1 \partial_v \varphi_2) \partial_x \psi \times \partial_y \psi - (\partial_u \varphi_2 \partial_v \varphi_1) \partial_x \psi \times \partial_y \psi \\
&= (\partial_u \varphi_1 \partial_v \varphi_2 - \partial_u \varphi_2 \partial_v \varphi_1) \partial_x \psi \times \partial_y \psi.
\end{aligned}$$

□

Exercise 2. The following function

$$\varphi\colon (0,1) \times (0,2\pi) \to \mathbb{R}^2, \quad \varphi(\rho,\vartheta) = (\rho\cos\vartheta, \rho\sin\vartheta)$$

is $C^1(\Omega;\mathbb{R}^2)$, where

$$\Omega = (0,1) \times (0,2\pi).$$

a) Show that

$$\varphi(\Omega) = U$$

where

$$B(O,1) - \{(x,0) \mid 0 \leq x < 1\};$$

b) show that $\varphi\colon \Omega \to U$ is a variable change, that is, φ is injective and

$$\partial_x \varphi \times \partial_y \varphi(x,y) \neq 0$$

for every $(x,y) \in \Omega$.

Solution. a) $\varphi(\Omega) \subseteq U$. We consider $(x,y) \in \varphi(\Omega)$. Then

$$(x,y) := \varphi(\rho,\vartheta)$$

for some $0 < \rho < 1$ and $0 < \vartheta < 2\pi$. Then

$$x^2 + y^2 = \rho^2 < 1.$$

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We have to show that $y \neq 0$. If $= 0$, then

$$\rho \sin \vartheta = 0.$$

Since $0 < \rho$,

$$\cos \vartheta = 0 \Rightarrow \vartheta = 2k\pi$$

for some $k \in \mathbb{Z}$. Since $0 < \vartheta < 2\pi$, this is not possible. Thus, $(x, y) \in U$.
$U \subseteq \varphi(\Omega)$. If $(x, y) \in U$, we have to find ρ and ϑ such that

$$(\rho \cos \vartheta, \rho \sin \vartheta) = (x, y).$$

We have

$$\rho = \sqrt{x^2 + y^2}$$

and $0 < \rho < 1$, because $(x, y) \in U$. Since $(x, y) \in U$, $(x, y) \neq (0, 0)$. Then, either $x \neq 0$ or $y \neq 0$. If $x \neq 0$, we have

$$\frac{\sin \vartheta}{\cos \vartheta} = \frac{y}{x} \Rightarrow \tan \vartheta = \frac{y}{x}.$$

Then there exists $\pi/2 < \vartheta < 3\pi/2$ such that

$$\vartheta = \arctan \frac{y}{x}.$$

If $y \neq 0$, we can write

$$\cot \vartheta = \frac{x}{y}.$$

The equation above has a solution in $0 < \vartheta < \pi$ or in $\pi < \vartheta < 2\pi$.

b) φ is injective. Let us consider two elements (ρ, ϑ) and (ρ', ϑ') such that

$$(\rho \cos \vartheta, \rho \sin \vartheta) = (\rho' \cos \vartheta', \rho' \sin \vartheta').$$

Then, taking the norm of the two vectors, we obtain

$$\rho = \rho'.$$

Consequently,

$$\sin \vartheta = \sin \vartheta', \quad \cos \vartheta = \cos \vartheta'$$

which implies $\vartheta = \vartheta'$ unless $\vartheta, \vartheta' \in \{0, 2\pi\}$. But this second case does not happen, because, by hypotheses

$$0 < \vartheta, \vartheta' < 2\pi.$$

Now,

$$\partial_\rho \varphi(\rho, \vartheta) = (\cos \vartheta, \sin \vartheta)$$
$$\partial_\vartheta \varphi(\rho, \vartheta) = (-\rho \sin \vartheta, \rho \cos \vartheta).$$

Then

$$\partial_\rho \varphi \times \partial_\vartheta \varphi = \rho \neq 0.$$

□

Exercise 3. We define the parametric surface

$$\psi \colon B(O, 1) \to \mathbb{R}^3, \quad \psi(x, y) = (x, y, x^2 - y^2)$$

a) is ψ injective?
b) evaluate $\psi_x \times \psi_y$
c) what is the area of ψ (the variable change of the second exercise can be helpful)?

Solution. (a) Yes, it is injective. Given (x, y) and (x', y') such that

$$\psi(x, y) = \psi(x', y')$$

there holds

$$(x, y, x^2 - y^2) = (x', y', x'^2 - y'^2) \Rightarrow x = x', y = y.$$

(b)

$$\partial_x \psi \times \partial_y \psi = (-2x, -2y, 1);$$

(c) the area of $\psi(B(O, 1))$ is given by

$$\iint_{B(O,1)} \sqrt{1 + 4x^2 + 4y^2}\, dxdy.$$

We use the variable change of the previous exercise. Therefore,

$$\begin{aligned}\iint_{B(O,1)} \sqrt{1 + 4x^2 + 4y^2}\, dxdy &= \iint_U \sqrt{1 + 4x^2 + 4y^2}\, dxdy \\ &= \int_0^{2\pi} \int_0^1 \sqrt{1 + 4\rho^2} \rho d\rho d\vartheta = 2\pi \int_0^1 \sqrt{1 + 4\rho^2} \rho d\rho d\vartheta \\ &= 2\pi \left[(1 + 4\rho^2)^{3/2} / 12 \right]_0^1 = \frac{(5^{3/2} - 1)\pi}{6}.\end{aligned}$$

□