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On the Orbital Stability of Standing-Wave Solutions to a Coupled Non-Linear Klein-Gordon Equation

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Abstract

We show the existence of standing-wave solutions to a coupled non-linear Klein-Gordon equation. Our solutions are obtained as minimizers of the energy under a two-charges constraint. We prove that the ground state is stable and that standing-waves are orbitally stable under a non-degeneracy assumption.

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1 Introduction

This work is on the orbital stability of standing-wave solutions

$$v_j(t,x) = e^{-i\omega_j t}u_j(x), \quad (t,x) \in \mathbb{R}\times\mathbb{R}^N, \quad 1 \le j \le 2 \tag{1.1}$$

to the coupled non-linear Klein-Gordon equation

$$\Box v_j + m_j^2 v_j + \partial_{z_j} F(v) = 0, \quad 1 \le j \le 2. \tag{CNLKG}$$

The m_j's are positive real numbers and

$$F\colon \mathbb{C}^2 \to \mathbb{C}$$

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is a continuously differentiable, real-valued function. Furthermore, we assume that $N \geq 3$ and

$$F(z) = -\mu|z_1 z_2|^\gamma + G(z), \quad 1 < \gamma < 1 + 2/N,\ \mu > 0, \tag{A_1}$$

$$|DG(z)| \leq c(|z|^{p-1} + |z|^{q-1}), \quad 2\gamma < p \leq q < 2^*,\ G(0) = 0, \tag{A_2}$$

$$G(z) = G(|z_1|, |z_2|), \quad G \geq 0, \tag{A_3}$$

$$\int_{\mathbb{R}^N} G(u_1^*, u_2^*) \leq \int_{\mathbb{R}^N} G(u_1, u_2),\ u_1, u_2 \geq 0, \tag{A_4}$$

$$V(z) := F(z) + \frac{1}{2}(m_1^2|z_1|^2 + m_2^2|z_2|^2) \geq 0,\ z \in \mathbb{R}^2. \tag{A_5}$$

Finally, we assume the local existence and uniqueness of strong solutions to (CNLKG) for initial data in $H^1 \times L^2$. By definition (check T. Tao [27, Remark 3.5, p. 126]), for every

$$\Phi := (\phi, \phi_t) \in H^1 \times L^2$$

there exists, uniquely, $T := T(\phi, \phi_t) > 0$ and

$$v_j \in C_t H_x^1([0,T) \times \mathbb{R}^N, \mathbb{C}) \cap C_t^1 L_x^2([0,T) \times \mathbb{R}^N, \mathbb{C}), \quad j = 1, 2$$

such that v solves (CNLKG) and

$$(v(0,\cdot), \partial_t v(0,\cdot)) = (\phi, \phi_t).$$

We also assume that local solutions can be extended to $\mathbb{R}$. We use the notation

$$U(t, \Phi) := (v(t,\cdot), \partial_t v(t,\cdot)) \in H^1 \times L^2.$$

In the scalar case, existence and local uniqueness of solutions to the non-linear Klein-Gordon equation with sub-critical growth condition has been addressed in [14, 7].

In assumption (A_4), u_j^* is the Steiner symmetrization. We refer to [18] for the definition and its properties. In the scalar case, (A_4) holds for every $G\colon \mathbb{R}^+ \to \mathbb{R}$. A simple example of G satisfying assumptions $(A_1$–$A_4)$ is given by

$$G(z) := |z|^p + |z|^q,$$

where q and p are as in (A_2). If m_j are large enough, then (A_5) also holds.

In the linear space

$$X := H^1(\mathbb{R}^N, \mathbb{C}^2) \times L^2(\mathbb{R}^N, \mathbb{C}^2)$$

we consider the metric induced by the following scalar product: given two vectors

$$\Phi = (\phi, \phi_t), \quad \Psi = (\psi, \psi_t),$$

we define

$$\langle \Phi, \Psi \rangle := \operatorname{Re} \sum_{j=1}^{2} \int_{\mathbb{R}^N} \left(\phi_j \overline{\psi}_j + D\phi_j \cdot \overline{D\psi_j} + \phi_j^t \overline{\psi_j^t} \right).$$

Definition 1.1 *A subset is stable if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$d(\Phi, S) < \delta \Rightarrow d(U(t, \Phi), S) < \varepsilon$$

for every $t \geq 0$.

Given $(z, w) \in \mathbb{C}^2 \times \mathbb{C}^2$, we define

$$(z \cdot w)_j := z_j w_j. \tag{1.2}$$

Following this notation, if v is a standing-wave as in (1.1), then

$$(v(0,\cdot), \partial_t v(0,\cdot)) = (u, -i\omega \cdot u).$$

Definition 1.2 *A standing-wave is orbitally stable if the subset of X*

$$\Gamma(u,\omega) = \left\{(\lambda \cdot u(\cdot + y), -i\omega \cdot \lambda \cdot u(\cdot + y)) \,|\, (\lambda, y) \in \mathbb{T}^2 \times \mathbb{R}^N\right\}$$

is stable.

From (A_3), if v is a standing-wave solution to (CNLKG), then (u,ω) is a solution to the elliptic system

$$-\Delta u_j + m_j^2 u_j + \partial_{z_j} F(u) = \omega_j^2 u_j,\ 1 \leq j \leq 2. \tag{1.3}$$

In order to solve (1.3), we follow the variational approach of [3], where the energy functional and the constraint are provided by conserved quantities: we refer to

$$X \ni (\phi, \phi_t) \mapsto \mathbf{E}(\phi, \phi_t) := \frac{1}{2}\sum_{j=1}^{2}\left(\int_{\mathbb{R}^N} |\phi_t^j|^2 + |D\phi_j|^2 + 2V(\phi)\right)$$

and

$$X \ni (\phi, \phi_t) \mapsto \mathbf{C}_j(\phi, \phi_t) := -\mathrm{Im}\int_{\mathbb{R}^N} \phi_t^j \overline{\phi}_j,\ 1 \leq j \leq 2$$

as *energy* and *charges*. By (A_3), the functions

$$\mathbb{R} \ni t \mapsto e(t) := \mathbf{E}(v(t,\cdot), \partial_t v(t,\cdot)), \tag{1.4}$$

$$\mathbb{R} \ni t \mapsto c_j(t) := \mathbf{C}_j(v(t,\cdot), \partial_t v(t,\cdot)) \tag{1.5}$$

are constant for every solution v. In particular, if v is a standing-wave as in (1.1), then

$$e(0) = \mathbf{E}(u, -i\omega\cdot u) = \frac{1}{2}\sum_{j=1}^{2}\int_{\mathbb{R}^N}\left(|Du_j|^2 + m_j^2 u_j^2 + \omega_j^2 u_j^2\right) + \int_{\mathbb{R}^N} F(u). \tag{1.6}$$

and

$$c_j(0) = \mathbf{C}_j(u, -i\omega\cdot u) = \omega_j \int_{\mathbb{R}^N} |u_j|^2. \tag{1.7}$$

We define the energy functional

$$E\colon H^1(\mathbb{R}^N, \mathbb{R}^2) \times \mathbb{R}^2 \to \mathbb{R}, \quad E(u,\omega) := \mathbf{E}(u, -i\omega\cdot u)$$

and the constraint

$$C_j\colon H^1(\mathbb{R}^N, \mathbb{R}^2) \times \mathbb{R}^2 \to \mathbb{R}, \quad C_j(u,\omega) := \omega_j \int_{\mathbb{R}^N} |u_j|^2$$

$$M_C := \{(u,\omega) \,|\, C_j(u,\omega) = C_j\}.$$

The key observation made in [3, Theorem 2.6] is that critical points of E constrained to M_C are classic solutions to (1.3). In Proposition 2.2 we prove this fact for the coupled case and that each of the components u_j does not change sign.

The main theorems of this work are the following:

Theorem 1.1 *Given a minimising sequence $(u_n, \omega_n)_{n\geq 1}$ for E over M_C, there exists a minimiser (u,ω) and $(y_n)_{n\geq 1} \subset \mathbb{R}^N$ such that, up to extract a subsequence,*

$$(u_n, \omega_n) = (u(\cdot + y_n), \omega) + o(1).$$

The proof is carried out by proving a concentration behaviour of the minimising sequences of the functional

$$J(u) = \frac{1}{2}\sum_{j=1}^{2}\int_{\mathbb{R}^N}|Du_j|^2 + \int_{\mathbb{R}^N}F(u)$$

on the constraint

$$N_\rho := \{u \mid \|u_j\|^2_{L^2(\mathbb{R}^N)} = \rho_j\}.$$

In turn, such behaviour follows from the sub-additivity property of the function $I(\rho) := \inf_{N_\rho} J$

$$I(\rho) < I(\tau) + I(\rho - \tau), \quad 0 < \tau_j \leq \rho_j,\ \tau \neq \rho. \tag{1.8}$$

Such property plays a crucial role in the proof of the orbital stability of standing-wave solutions to a variety of evolution problems: the non-linear Schrödinger equation, [11, 4], coupled NLS in dimension $N = 1$, in [21], and KdV-NLS systems [1]. In these references, (1.8) is obtained through rescaling argument (as in [4]) or symmetries arising from the choice of the non-linear term (as in [21]). Due to the lack of suitable rescaling arguments for non-linearities satisfying (A_1), we obtain (1.8) from considerations on the gradient terms. We exploit an idea carried out by J. Byeon in [10, Proposition 1.4] which is based on the symmetric rearrangement and we prove that, if

$$(u, v) \in N_\tau \times N_{\rho-\tau}$$

have disjoint support and are a good approximation of $I(\tau)$ and $I(\rho - \tau)$, respectively, then there exists $D = D(\rho, \tau) > 0$ such that

$$\|Dw^*\|^2 \leq \|Du\|^2 + \|Dv\|^2 - D,$$

where $w = u + v$ and w^* is the Steiner symmetrization of w. We prove this fact in Lemma 3.1.

Preliminary notation is required to introduce the next results. To C in $\mathbb{R}^2$, we can associate the subset

$$m_C := \inf_{M_C} E, \quad K_C := \{(u, \omega) \in M_C \mid E(u, \omega) = m_C\}$$

and

$$\Gamma_C = \bigcup_{(u,\omega)\in K_C} \Gamma(u, \omega).$$

Theorem 1.2 *Let $C \in \mathbb{R}^2$ be such that $C_j \neq 0$ for $j = 1, 2$. Given a sequence*

$$(\Phi_n)_{n\geq 1} \subset X$$

then $d(\Phi_n, \Gamma_C) \to 0$ if and only if

$$\mathbf{E}(\Phi_n) \to m_C \text{ and } \mathbf{C}_j(\Phi_n) \to C_j.$$

A proof of this theorem in the scalar case can be found in [3, §3.1]. We included a proof which does not use the local well-posedness of (CNLKG) (implicitly used in [3, Lemma 3.5]). Our proof relies on an improved version of the Convexity Inequality for Gradients, [18, Theorem 7.8, p., 177], outlined in Lemma 5.1. Theorem 1.2 implies that

$$X \ni \Phi \mapsto \mathbf{V}(\Phi) := (\mathbf{E}(\Phi) - m_C)^2 + \sum_{j=1}^{2}(\mathbf{C}_j(\Phi) - C_j)^2 \tag{1.9}$$

is a Lyapunov function for Γ_C (see [3, Definition 2.4]).

Given a subset $S \subset H^1 \times \mathbb{R}^2$ and (u,ω) in K_C, we define the following subsets of $H^1 \times \mathbb{R}^2$:

$$B_\delta(S) := \{(w,\alpha) \,|\, d((w,\alpha), S) < \delta\}, \quad G(u,\omega) := \{(u(\cdot + y), \omega) \,|\, y \in \mathbb{R}^N\}.$$

We say that (u,ω) satisfies the condition (D) if there exists $\delta > 0$ such that, for every

$$(u', \omega') \in K_C \setminus \{(u,\omega)\}$$

such that

$$\Gamma(u', \omega') \neq \Gamma(u, \omega),$$

there holds

$$B_\delta(G(u,\omega)) \cap G(u', \omega') = \emptyset. \tag{D}$$

Theorem 1.3 *The subset $\Gamma_C \subset X$ is stable. For every minimiser (u,ω) fulfilling condition* (D), *$\Gamma(u,\omega)$ is stable.*

We intentionally restricted our work to the higher dimensional case $N \geq 3$ and to $C_1C_2 \neq 0$. We address to further works the treatment of the semi-trivial case ($C_j = 0$ for some $1 \leq j \leq 2$), and the lower dimensions $N = 1, 2$.

Results on the orbital stability of standing-wave solutions to coupled non-linear Klein-Gordon equations, with a different variational characterisation, have been obtained in [30]. Numerical results on the existence of coupled standing-waves have been obtained in [8] when $N = 3$ and the non-linearity has a critical exponent.

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2 Basic properties of the functional J

Proposition 2.1 *For every ρ in $\mathbb{R}^2$ with $\rho_j > 0$,*

(i) J attains negative values on N_ρ;

(ii) J is bounded from below and minimising sequences of J over N_ρ are bounded;

moreover,

(iii) J is continuous;

(iv) given a weakly converging sequence $u_n \rightharpoonup u$, up to extract a subsequence

$$J(u_n - u) = J(u_n) - J(u) + o(1).$$

Proof. (i) By choosing a test function in a neighbourhood of the origin we can write $F = F_0 + F_\infty$, where

$$|F_0(z)| \leq c|z|^p, \quad |F_\infty(z)| \leq c|z|^q.$$

A sequence $(u_n)_{n\geq 1}$ such that $u_n \to u$ in H^1, converges in $L^p(\mathbb{R}^N)$ by the Sobolev inequality

$$\|u\|_{L^p} \leq S\,\|u\|_{L^2}^{1-\frac{N}{2}+\frac{N}{p}} \|Du\|_{L^2}^{\frac{N}{2}-\frac{N}{p}}. \tag{3.1}$$

There exists g in $L^p(\mathbb{R}^N)$ and a subsequence $(u_{n_k})_{k\geq 1}$ such that

$$|u^j_{n_k}| \leq g$$

and $u_{n_k} \to u$ pointwise a.e., by [6, Théorème IV.9, p. 58]. Then

$$\int_{\mathbb{R}^N} F_0(u_{n_k}) \to \int_{\mathbb{R}^N} F_0(u)$$

by the dominated convergence theorem. We can extract a subsequence alike from every subsequence of $(u_n)_{n\geq 1}$. Then, the map $u \mapsto \int F_0 \circ u$ is continuous. Similarly, $u \mapsto \int F_\infty \circ u$ is continuous, and the gradient part of J is smooth. An adaptation of the technique used in [2, Theorem 2.6, p. 17] would allow to conclude that J is $C^1(H^1, \mathbb{R})$.
(ii) We refer to Step II of the Appendix of [4], which addresses the scalar case.
(iii) Following the proof of [4, Lemma 5], we can show that J attains negative values on N_ρ for every choice of ρ: setting

$$\lambda := (\rho_1^{-1}\rho_2)^{1/2}$$

and

$$u := (w, \lambda w),\ \ w \in N_{\rho_1},$$

we have

$$J(u) = (1 + \lambda^2)^{-1} J_1(w),$$

where

$$J_1(w) := \frac{1}{2}\int_{\mathbb{R}^N} |Dw|^2 + \int_{\mathbb{R}^N} F_1(w)$$
$$F_1(s) := \left(1 + \lambda^2\right)^{-1}\left(-\mu\lambda^\gamma |s|^{2\gamma} + G(s, \lambda s)\right).$$

By (A_1) and (A_2) the non-linearity F_1 fulfils hypotheses (F_p) and (F_2) of [4]. Then, by [4, Lemma 5], there exists w such that $J_1(w) < 0$. Then $J(u) < 0$.
(iv) By the Hölder inequality and (A_3), we have

$$\begin{aligned} J(u) &\geq \frac{1}{2}\sum_{j=1}^{2} \|Du_j\|^2_{L^2} - 2\mu\left(\|u_1\|_{L^{2\gamma}}\|u_2\|_{L^{2\gamma}}\right)^\gamma \\ &\geq \frac{1}{2}\sum_{j=1}^{2}\left(\|Du_j\|^2_{L^{2\gamma}} - \mu\|u_j\|^{2\gamma}_{L^{2\gamma}}\right). \end{aligned} \tag{3.2}$$

From (3.1), there exists a constant c' such that

$$J(u) \geq c' \sum_{j=1}^{2} \|Du_j\|^2_{L^2} - \|Du_j\|^{2\gamma\left(\frac{N}{2} - \frac{N}{2\gamma}\right)}_{L^2}. \tag{3.3}$$

By the hypotheses on γ in (A_1), the right member of the above inequality is bounded from below, as J is. Given a minimising sequence, $(u_n)_{n\geq 1}$ in N_ρ, for n large we have $J(u_n) < 0$, by (i). Then, $\|Du_n\|_{L^2}$ is bounded by (3.3). Because $\|u^j_n\|^2_{L^2} = \rho_j$, the H^1-norm is bounded too.

Proposition 2.2 *Given C in $\mathbb{R}^2$ such that $C_1C_2 \neq 0$, the following properties hold:*

(i) E is coercive;

(ii) critical points of E over M_C are solutions to the elliptic system (1.3)*;*

(iii) if (u, ω) is a minimiser, then for $j = 1, 2$, u_j is either positive or negative.

Proof. The proof of (i) follows from the arguments used in *Step I* of [3, Proof of Lemma 2.7].
(ii) If (u, ω) is a critical point, there are two Lagrange multipliers λ_1 and λ_2 such that

$$DE = \lambda_1 DC_1 + \lambda_2 DC_2.$$

Taking the projection on $H^1(\mathbb{R}^N, \mathbb{R}^2) \times \{0\}$, and on $\{0\} \times \mathbb{R}^2$, we obtain

$$-\Delta u_j + m_j^2 u_j + \partial_{z_j} F(u) = 2\lambda_j \omega_j u_j,$$
$$\omega_j \|u_j\|_{L^2}^2 = \lambda_j \|u_j\|_{L^2}^2$$

for $j = 1, 2$. Because $u_j \neq 0$ we obtain $\lambda_j = \omega_j$ and thus (1.3). By local regularity results, [13, §8], u is a classic solution.
(iii) We define

$$w_j := |u_j| \geq 0.$$

From (A_3) it follows that $(w, \omega) \in M_C$ and $E(u, \omega) = E(w, \omega)$. By (ii),

$$-\Delta w_j = (\omega_j^2 - m_j^2) w_j + \gamma \mu w_j^{\gamma - 1} w_{\sigma(j)}^{\gamma} - \partial_{z_j} G(w)$$

where $\sigma(1) = 2$ and $\sigma(2) = 1$. Hence,

$$-\Delta w_j + \lambda_j w_j + \partial_{z_j} G(w) \geq 0,$$

where $\lambda_j := m_j^2 - \omega_j^2$. Let us define

$$A_j(x) = \begin{cases} \lambda_j + \partial_{z_j} G(w) w_j^{-1} & \text{if } w_j(x) \neq 0 \\ \lambda_j & \text{otherwise.} \end{cases}$$

By (A_2), and the continuity of w_j and $\partial_{z_j} G$, we have A_j^+ is $C_+(\mathbb{R}^N)$. Therefore,

$$-\Delta w_j + A_j^+(x) w_j \geq 0.$$

Now, we can apply the strong maximum principle. Hence, $w_j > 0$.

3 The sub-additivity property of I

Given a non-negative function f, we denote with f^{*e} the Steiner symmetrization with respect to the direction e in $\mathbb{R}^N$ (with $|e| = 1$), [18, §3.7, p. 87]. We denote with f^* the symmetric rearrangement, [18, §3.3, p. 80]. The next lemma addresses the one-dimensional case of [10, Proposition 1.4]. The argument goes back to B. Kawohl [17, Lemma 2.6, p. 33].

Lemma 3.1 *Let u, v be $H^1(\mathbb{R})$ non-negative functions with compact support, symmetric and radially decreasing with respect to the origin, and such that $u(0) \leq v(0)$. Let T be such that*

$$\text{supp}\,(u) \cap \text{supp}\,(v(\cdot - T)) = \emptyset.$$

Then

$$\|w^{*\prime}\|_{L^2}^2 \leq \|w'\|_{L^2}^2 - \frac{3}{4}\|u'\|_{L^2}^2$$

where $w(t) := u(t) + v(t - T)$.

Proof. We denote with $[-c, c]$ and $[-d, d]$ the support of u and v, respectively. Firstly, we prove the estimate under the additional assumptions that u and v are continuously differentiable and

$$tu'(t) < 0 \text{ on } \{t \in (-c, c), t \neq 0\} \tag{5.1}$$

$$tv'(t) < 0 \text{ on } \{t \in (-d, d), t \neq 0\}. \tag{5.2}$$

We set $a := \sup(u)$ and $b := \sup(v)$. The functions

$$u\colon [0, c] \to [0, a], \quad v\colon [0, d] \to [0, b]$$

are invertible because they are strictly decreasing. Their inverses, y_u and y_v, are continuously differentiable on $(0, a)$ and $(0, b)$, respectively. Thus,

$$u(y_u(s)) = s \text{ on } [0, a], \quad v(y_v(s)) = s \text{ on } [0, b]. \tag{5.3}$$

Because w^* is symmetric and decreasing, the level set $\{w^* \geq s\}$ is an interval. We denote its length by $2z(s)$. We have

$$2z(s) = |\{w^* \geq s\}| = \begin{cases} 2y_u(s) + 2y_v(s) & \text{if } s \in [0, a] \\ 2y_v(s) & \text{if } s \in [a, b]. \end{cases} \tag{5.4}$$

Thus, z is strictly decreasing and continuously differentiable for every $s \notin \{0, a, b\}$. Moreover,

$$w^*(z(s)) = s \text{ on } [0, b]. \tag{5.5}$$

Taking the derivative with respect to s in (5.5) and in (5.3), we have

$$w^{*\prime}(z(s))z'(s) = 1, \quad u'(y_u(s))y_u'(s) = 1, \quad v'(y_v(s))y_v'(s) = 1 \tag{5.6}$$

on the complement of a finite set. Hence,

$$\begin{aligned} \int_{\mathbb{R}} |w^{*\prime}|^2 dt &= 2\int_0^{c+d} |w^{*\prime}|^2 dt = -2\int_0^b |w^{*\prime}(z(s))|^2 z'(s) ds = -2\int_0^b (z'(s))^{-1} ds \\ &= -2\int_0^a (y_u'(s) + y_v'(s))^{-1} ds - 2\int_a^b (y_v'(s))^{-1} ds. \end{aligned} \tag{5.7}$$

The second equality follows from a change of variable and the first of (5.6). The fourth equality follows from (5.4). From the inequality

$$2(x + y)^{-1} \leq x^{-1} + y^{-1} - \max\{x^{-1}, y^{-1}\}, \quad x, y > 0$$

the first integration of the second line of (5.7) can be estimated from above with

$$-\int_0^a \Big((y_u'(s))^{-1} + (y_v'(s))^{-1}\Big) ds + \int_0^a \max\{y_u'(s)^{-1}, y_v'(s)^{-1}\} ds. \tag{5.8}$$

Using the estimate $2\max\{t, s\} \geq t + s$, (5.8) and (5.7), the left member of the first equality in (5.7) is bounded by

$$\begin{aligned} &-\frac{1}{2}\int_0^a (y_u'(s))^{-1} ds - \frac{1}{2}\int_0^a (y_v'(s))^{-1} ds - 2\int_a^b (y_v'(s))^{-1} ds \\ \leq &-\frac{1}{2}\int_0^a (y_u'(s))^{-1} ds - 2\int_0^b (y_v'(s))^{-1} ds \\ = &\frac{1}{4} \cdot \left(-2\int_0^a (y_u'(s))^{-1}\right) + \left(-2\int_0^b (y_v'(s))^{-1}\right) ds. \end{aligned}$$

From a change of variable and (5.6), it follows that

$$\|u'\|_{L^2}^2 = -2\int_0^a (y_u'(s))^{-1}ds, \quad \|v'\|_{L^2}^2 = -2\int_0^b (y_v'(s))^{-1}ds.$$

Thus, from (5.7), we obtain

$$\|w^{*\prime}\|_{L^2}^2 \le \frac{1}{4}\|u'\|_{L^2}^2 + \|v'\|_{L^2}^2 = \|w'\|_{L^2}^2 - \frac{3}{4}\|u'\|_{L^2}^2. \tag{5.9}$$

In the general case, we can approximate u and v with functions satisfying (5.1) and (5.2): firstly, we consider

$$\sigma_u\colon [0,c] \to \mathbb{R}^+, \quad \sigma_u'(t) < 0 \text{ on } (0,c), \quad \sigma_u'(0) = 0 \tag{5.10}$$

smooth, and extend it to $\mathbb{R}$ as $\sigma_u(-t) = \sigma_u(t)$. We define

$$U := u + \|u - v\|_{L^\infty(0,\delta)}\, \sigma_u, \; u_\delta := \rho_\delta * U \tag{5.11}$$

where ρ_δ is a symmetric mollifier. Thus, u_δ is an even function. Because U is strictly decreasing, given $t \ge 0$, we have

$$u_\delta'(t) = \int_0^\delta \rho_\delta'(y)(U(t-y) - U(t+y))dy < 0,$$

unless $t = 0$. Similarly, we define σ_v as in (5.10) with the additional hypothesis

$$\sigma_u(0) < \sigma_v(0) - 1.$$

V and v_δ are defined as in (5.11), by replacing σ_u with σ_v. Thus, if $\delta > 0$ is sufficiently small,

$$\sup(u_\delta) \le \sup(v_\delta)$$

and the supports of u_δ and $v_\delta(\cdot - T)$ are disjoint. Therefore, we can apply estimate (5.9) to

$$w_\delta = u_\delta + v_\delta(\cdot - T)$$

and obtain

$$\|w_\delta^{*\prime}\|_{L^2}^2 \le \|w_\delta'\|_{L^2}^2 - \frac{3}{4}\|u_\delta'\|_{L^2}^2.$$

By the continuity of the symmetric rearrangement in $H^1(\mathbb{R})$, [12], we can take the limit as $\delta \to 0$ in the above inequality.

Proposition 3.1 *Let ρ, τ be such that $\rho_j \ge \tau_j > 0$ and $\tau \ne \rho$. Then,*

$$I(\rho) < I(\tau) + I(\rho - \tau).$$

Proof. Define $\sigma := \rho - \tau$, and let

$$(u_n)_{n\ge1} \subset N_\tau, \quad (v_n)_{n\ge1} \subset N_\sigma \tag{5.12}$$

be minimising sequences of J over N_τ and N_σ, respectively. By (iii) of Proposition 2.1, we can suppose that each of the sequences have compact support, that u_n^j and v_n^j are non-negative, from (A_3), and symmetrically decreasing, by (A_1), (A_4), [26, Lemma 1] and [18, Theorem 3.4, p. 82].

We set $e_N := (0, \dots, 0, 1)$. Let $(T_n)_{n\ge1}$ be a real sequence such that the two functions

$$u_n^i, v_n^j(\cdot + T_n e_N)$$

have disjoint support for every i, j in $\{1,2\}$. Then,

$$w_n := u_n + v_n(\cdot + T_n e_N) \in N_\rho \tag{5.13}$$

$$J(w_n) = J(u_n) + J(v_n). \tag{5.14}$$

We denote the Steiner symmetrization of w_n with respect to e_N with $w_n^{*e_N}$. By [17, (C), p. 22], $w_n^{*e_N} \in N_\rho$. From [18, (v), p. 81], and [18, Eq. (1), p. 82],

$$-\int_{\mathbb{R}^N} |w_n^{1*e_N} w_n^{2*e_N}|^\gamma dx \leq -\int_{\mathbb{R}^N} |w_n^1 w_n^2|^\gamma dx.$$

Along with (A_4), the above inequality yields

$$\int_{\mathbb{R}^N} F(w_n^{*e_N}) \leq \int_{\mathbb{R}^N} F(w_n).$$

By [26, Lemma 1],

$$\|Dw_n^{j*e_N}\|_{L^2} \leq \|Dw_n^j\|_{L^2}. \tag{5.15}$$

Thus $J(w_n^{*e_N}) \leq J(w_n)$. Given $x' \in \mathbb{R}^{N-1}$,

$$\partial_{x_N} w_n^{j*e_N}(x', t) = w_n^{j*}(x', \cdot)'(t).$$

Then, we can write

$$\begin{aligned}\int_{\mathbb{R}^N} |\partial_{x_N} w_n^{j*e_N}|^2 dx &= \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}} |w_n^{j*}(x', \cdot)'(t)|^2 dt\, dx' \\ &= \int_{U_n^j} \int_{\mathbb{R}} |w_n^{j*}(x', \cdot)'(t)|^2 dt\, dx' \\ &+ \int_{V_n^j} \int_{\mathbb{R}} |w_n^{j*}(x', \cdot)'(t)|^2 dt\, dx' =: A_1^j + A_2^j\end{aligned} \tag{5.16}$$

where

$$U_n^j = \{x' \in \mathbb{R}^{N-1} \mid \sup_{\mathbb{R}} u_n^j(x', \cdot) \leq \sup_{\mathbb{R}} v_n^j(x', \cdot)\}$$
$$V_n^j = \{x' \in \mathbb{R}^{N-1} \mid \sup_{\mathbb{R}} v_n^j(x', \cdot) < \sup_{\mathbb{R}} u_n^j(x', \cdot)\}.$$

For every $x' \in \mathbb{R}^{N-1}$, $u_n^j(x', \cdot)$ and $v_n^j(x', \cdot)$ satisfy the hypotheses of Lemma 3.1 with $T = T_n$. Thus,

$$A_1^j \leq \int_{U_n^j} \Big(\|w_n^j(x', \cdot)'\|_{L^2(\mathbb{R})}^2 - \frac{3}{4}\|u_n^j(x', \cdot)'\|_{L^2(\mathbb{R})}^2\Big) dx'$$
$$A_2^j \leq \int_{V_n^j} \Big(\|w_n^j(x', \cdot)'\|_{L^2(\mathbb{R})}^2 - \frac{3}{4}\|v_n^j(x', \cdot)'\|_{L^2(\mathbb{R})}^2\Big) dx'.$$

Taking the sum, we obtain

$$\begin{aligned}A_1^j + A_2^j &\leq \|\partial_{x_N} w_n^j\|_{L^2}^2 \\ &\quad - \frac{3}{4}\left(\|\partial_{x_N} u_n^j\|_{L^2(U_n^j \times \mathbb{R})}^2 + \|\partial_{x_N} v_n^j\|_{L^2(V_n^j \times \mathbb{R})}^2\right).\end{aligned}$$

Because u_n^j and $|\partial_{x_i} u^j|$ are radially symmetric, we have

$$\|Du_n^j\|_{L^2(U_n^j \times \mathbb{R})}^2 = N\|\partial_{x_N} u_n^j\|_{L^2(U_n^j \times \mathbb{R})}^2.$$

From (5.13), it follows that

$$N\|\partial_{x_N} w_n^j\|_{L^2}^2 = \|Dw_n^j\|_{L^2}^2.$$

Thus,

$$N(A_1^j + A_2^j) \le \|Dw_n^j\|_{L^2}^2 - \frac{3}{4}\left(\|Du_n^j\|^2_{L^2(U_n^j\times\mathbb{R})} + \|Dv_n^j\|^2_{L^2(V_n^j\times\mathbb{R})}\right). \tag{5.17}$$

We define

$$d_n^j = \|Du_n^j\|^2_{L^2(U_n^j\times\mathbb{R})} + \|Dv_n^j\|^2_{L^2(V_n^j\times\mathbb{R})}.$$

We prove that $(d_n^j)_{n\ge1}$ is bounded from below. On the contrary, up to extract a subsequence, we can suppose that $d_n^j \to 0$ for some $1 \le j \le 2$. Because u_n and v_n are minimising sequences, by (ii) of Proposition 2.1, they are also bounded in H^1. By construction, u_n and v_n are radially decreasing. Then, by [5, Theorem A.I'], up to extract a subsequence, we can suppose that

$$u_n^j \to u_j, v_n^j \to v_j \text{ in } L^{2\gamma}(\mathbb{R}^N), \text{ a.e.}$$

By (i) of Proposition 2.1 and the first inequality in (3.2)

$$\|u_n^j\|_{L^{2\gamma}}, \|v_n^j\|_{L^{2\gamma}} \ge c = c(\rho,\tau) > 0, \tag{5.18}$$

whence $u_j, v_j \not\equiv 0$. We fix $R > 0$ and consider the domains

$$E_n^j := (U_n^j \times \mathbb{R}) \cap B_R, \quad F_n^j := (V_n^j \times \mathbb{R}) \cap B_R. \tag{5.19}$$

Because the two domains are bounded,

$$d_n^j \ge \frac{1}{m(E_n^j)} \cdot \|Du_n\|^2_{L^1(E_n^j)} + \frac{1}{m(F_n^j)} \cdot \|Dv_n\|^2_{L^1(F_n^j)} \ge \frac{1}{\omega_N R^N}\left(\|Du_n^j\|^2_{L^1(E_n^j)} + \|Dv_n^j\|^2_{L^1(F_n^j)}\right). \tag{5.20}$$

Up to extract a subsequence there are two sets $U_j, V_j \subset \mathbb{R}^{N-1}$ such that the convergence

$$\chi_{U_n^j} \to \chi_{U_j}, \quad \chi_{V_n^j} \to \chi_{V_j}$$

is strong in $L^2(B_R^{N-1})$, where $B_R^{N-1} := B_R \cap (\mathbb{R}^{N-1} \times 0)$. Moreover, U_j and V_j are radially symmetric and the convergence

$$\chi_{E_n^j} \to \chi_{E_j}, \quad \chi_{F_n^j} \to \chi_{F_j}$$

is strong in $L^2(B_R)$, where

$$E_j = (U_j \times \mathbb{R}) \cap B_R, \quad F_j = (V_j \times \mathbb{R}) \cap B_R.$$

Taking the limit in (5.20), we obtain

$$Du_j \equiv 0, \ E_j \text{ a.e.}, \quad Dv_j \equiv 0, \text{ on } F_j \text{ a.e.}$$

whence

$$Du_j \equiv 0 \text{ on } U_j, \quad Dv_j \equiv 0 \text{ on } V_j \tag{5.21}$$

and

$$u_j \le v_j \text{ on } U_j, \quad v_j \le u_j \text{ on } V_j. \tag{5.22}$$

By the Ekeland Principle, we can suppose that the sequences in (5.12) are Palais-Smale. Therefore, u_j and v_j are weak solutions to an elliptic system and, by local regularity results, continuously differentiable. Thus, we can suppose that U_j is open and V_j is closed. Because such sets are radially symmetric, we can write

$$U_j = \{x' \in B_R^{N-1} \,|\, |x'| \in \Omega\}, \quad V_j = \{x' \in B_R^{N-1} \,|\, |x'| \in G\}$$

where Ω and G are open and closed subsets of $\langle e_1 \rangle$. We set

$$\Omega_1 := \Omega \cap \{te_1 \,|\, t > 0\}, \quad G_1 := G \cap \{te_1 \,|\, t > 0\}.$$

Then

$$\Omega_1 = \bigcup_{i\in\mathbb{Z}} (a_i, b_i),\ a_i \le b_i, \quad G_1 = \bigcup_{i\in\mathbb{Z}} [b_i, a_{i+1}].$$

For every $i \in \mathbb{Z}$, v_j is constant on $[b_i, a_{i+1}]$ by (5.21). Thus,

$$v_j(b_i) = v_j(a_{i+1}). \tag{5.23}$$

In the case $b_i = a_{i+1}$ the above equality is obviously true. By the continuity of u_j and v_j, and (5.22) and (5.21), it follows

$$u_j(b_i) = v_j(b_i), \quad u_j(a_{i+1}) = v_j(a_{i+1}) \tag{5.24}$$

$$u_j \equiv c_i \text{ on } (a_i, b_i) \tag{5.25}$$

for some constant $c_i \in \mathbb{R}$. From (5.23) and (5.24) we have

$$c_i = u_j(b_i) = v_j(b_i) = v_j(a_{i+1}) = u_j(a_{i+1}) = c_{i+1}.$$

Given $x \in [b_i, a_{i+1}]$

$$c_i \ge u_j(x) \ge c_{i+1} = c_i,$$

because u_j is monotonically non-increasing. Then, u_j is constant on $\{te_1 \,|\, t > 0\}$. Because u_j is radially symmetric, u_j is constant on B_R. By applying the same argument for every $R > 0$, we obtain that u_j is constant on $\mathbb{R}^N$. Because u_j is L^2, we have $u_j \equiv 0$ obtaining a contradiction with (5.18). The contradiction follows from the assumption that $d_n^j \to 0$. So, we proved that each of the sequences $(d_n^j)_{n\ge1}$ is bounded from away from zero. Let d be such that

$$d_n^j \ge d \text{ for all } n.$$

Then, from (5.16), (5.17) we obtain

$$N \int_{\mathbb{R}^N} |\partial_{x_N} w_n^{j*e_N}|^2 dx \le \|Dw_n^j\|_{L^2}^2 - \frac{3d_n^j}{4} \le \|Dw_n^j\|_{L^2}^2 - \frac{3d}{4}. \tag{5.26}$$

Finally, we consider the decreasing rearrangement of $w_n^{j*e_N}$. By applying (5.15) in dimension $N = 1$, we have

$$\begin{aligned}\|\partial_{x_N} w_n^{j*e_N*}\|_{L^2}^2 &= \int_{\mathbb{R}^{N-1}} \|w_n^{j*e_N*}(x', \cdot)'\|_{L^2(\mathbb{R})}^2 dx' \\ &\le \int_{\mathbb{R}^{N-1}} \|w_n^{j*e_N}(x', \cdot)'\|_{L^2(\mathbb{R})}^2 dx' = \|\partial_{x_N} w_n^{j*e_N}\|_{L^2}^2.\end{aligned}$$

From (5.26), we note

$$N \int_{\mathbb{R}^N} |\partial_{x_N} w_n^{j*e_N*}|^2 \le \|Du_n^j\|_{L^2}^2 + \|Dv_n^j\|_{L^2}^2 - \frac{3d}{4}.$$

Because $w_n^{j*e_N*}$ is radially symmetric, from (5.26) it follows that

$$\int_{\mathbb{R}^N} |Dw_n^{j*e_N*}|^2 \leq \|Du_n^j\|_{L^2}^2 + \|Dv_n^j\|_{L^2}^2 - \frac{3d}{4}$$

and

$$J(w_n^{*e_N*}) \leq J(w_n^{*e_N}),\ w_n^{*e_N*} \in N_\rho.$$

Hence,

$$I(\rho) \leq J(w_n^{*e_N*}) \leq J(u_n) + J(v_n) - \frac{3d}{4}$$

Taking the limit as $n \to \infty$, we obtain

$$I(\rho) \leq I(\tau) + I(\sigma) - \frac{3d}{4}.$$

We set $D := 3d/4 > 0$.

4 Minimising sequences of (J, N_ρ) and (E, M_C)

Lemma 4.1 *Let $(u_n)_{n\geq 1}$ be a bounded sequence in H^1 such that*

$$\liminf_{n\to\infty} \int_{\mathbb{R}^N} |u_n^1 u_n^2|^\gamma > 0$$

where $1 < \gamma < 2^*/2$. Then, there exist $u \in H^1$ and a sequence $(y_n)_{n\geq 1} \subset \mathbb{R}^N$ such that

$$u_n^j(\cdot - y_n) \rightharpoonup u_j, \quad u_1 u_2 \not\equiv 0.$$

Proof. Let $w_n = u_n^1 u_n^2$. From the Schwarz inequality, we have

$$w_n \in L^1(\mathbb{R}^N);$$

by applying the Hölder inequality with the pair of exponents

$$\left(\frac{2(N-1)}{N}, \frac{2(N-1)}{N-2}\right),$$

we obtain

$$Dw_n \in L^{N/(N-1)}(\mathbb{R}^N).$$

We use [20, Lemma I.1] with $q = 1$ and $p = N/(N-1)$. Hence, given $R > 0$, either there exists a sequence $(y_n)_{n\geq 1}$ such that

$$\liminf_{n\to\infty} \int_{B(-y_n,R)} |w_n| > 0 \tag{6.1}$$

or

$$w_n \to 0 \text{ in } L^\alpha(\mathbb{R}^N),\ \alpha \in (1, N/(N-2)).$$

The latter is ruled out by the hypothesis on γ. Hence, (6.1) holds. We set

$$v_n^j := u_n^j(\cdot - y_n)$$

and obtain

$$\liminf_{n\to\infty} \int_{B_R} |v_n^1 v_n^2| > 0. \tag{6.2}$$

Because v_n^j are bounded in H^1, we can suppose that they converge weakly to some limits u_1 and u_2, respectively. By the Rellich-Kondrakhov Theorem, we can suppose that such convergence is strong in $L^2(B_R)$. Thus, (6.2) yields

$$\int_{B_R} u_1 u_2 > 0$$

which implies $u_1 u_2 \not\equiv 0$.

Theorem 4.1 *Let $(u_n)_{n\geq 1}$ be a minimising sequence for J over N_ρ. Then, there exists $u \in N_\rho$ and a sequence $(y_n)_{n\geq 1}$ such that*

$$u_n = u(\cdot + y_n) + o(1) \text{ in } H^1$$
$$J(u) = \inf_{N_\rho} J.$$

Proof. By (i) and (ii) of Proposition 2.1, $I(\rho) < 0$ and the sequence $(u_n)_{n\geq 1}$ is bounded. Because $G \geq 0$, $(u_n)_{n\geq 1}$ fulfils the hypothesis of Lemma 4.1 if $\gamma < N/(N-2)$ holds. This, in turn, follows from (A_1) and

$$1 + \frac{2}{N} < \frac{N}{N-2}.$$

Then, we consider the sequence $(y_n)_{n\geq 1}$ and $u \in H^1$ given by Lemma 4.1. We define

$$v_n := u_n(\cdot - y_n) - u, \quad \tau := (\|u_1\|_{L^2}^2, \|u_2\|_{L^2}^2).$$

Note that $\tau_j \leq \rho_j$ by the weak lower semi-continuity property of the L^2-norm and that $\tau_j > 0$, from Lemma 4.1. Suppose that $\tau \neq \rho$. By (iv) of Proposition 2.1, up to extract a subsequence, we can suppose that

$$J(v_n) = J(u_n(\cdot - y_n)) - J(u) + o(1).$$

After a change of variable, the first term of the right member equals $J(u_n)$, which converges to $I(\rho)$. Hence, by Proposition 3.1

$$I(\rho - \tau) \leq I(\rho) - I(\tau) < I(\rho - \tau).$$

Thus, we obtain a contradiction with the assumption that $\tau \neq \rho$. Then $\tau = \rho$ and $u \in N_\rho$. Thus,

$$u_n^j(\cdot - y_n) - u_j \to 0 \text{ in } L^2(\mathbb{R}^N).$$

Up to extract a subsequence, we can suppose that the above convergence is weak in H^1. We set $w_n := u_n(\cdot - y_n)$. By (3.1), the above convergence holds in $L^p(\mathbb{R}^N)$ and $L^q(\mathbb{R}^N)$. Therefore, as in the proof of (iii) of Proposition 2.1, we conclude that

$$\int_{\mathbb{R}^N} F(w_n) \to \int_{\mathbb{R}^N} F(u).$$

We have

$$J(w_n) = \int_{\mathbb{R}^N} F(w_n) + \frac{1}{2}\int_{\mathbb{R}^N} |Dw_n|^2 \geq \int_{\mathbb{R}^N} F(u) + \frac{1}{2}\int_{\mathbb{R}^N} |Du|^2 = J(u).$$

Because $(w_n)_{n\geq 1}$ is a minimising sequence, taking the limit, we obtain

$$I(\rho) = \int_{\mathbb{R}^N} F(u) + \lim_{n\to\infty} \frac{1}{2}\int_{\mathbb{R}^N} |Dw_n|^2 \geq \int_{\mathbb{R}^N} F(u) + \frac{1}{2}\int_{\mathbb{R}^N} |Du|^2 = J(u) \geq I(\rho).$$

Then, the two above inequalities are equalities:

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |Dw_n|^2 = \int_{\mathbb{R}^N} |Du|^2, \; J(u) = I(\rho).$$

Thus, $Dw_n \to Du$ strongly in L^2 and u is a minimiser.

Proof of Theorem 1.1 From (i) of Proposition 2.2, given a minimising sequence (u_n, ω_n), there exists ρ such that

$$\|u_n^j\|_{L^2} \to \sqrt{\rho_j} > 0, \quad \omega_n \to \omega.$$

As in *Step II* of the proof of [3, Lemma 2.7], it can be shown that

$$v_n^j = \frac{\sqrt{\rho_j} u_n^j}{\|u_n^j\|_{L^2}}$$

is a minimising sequence for J over N_ρ (notice that, unlike stated in [3, p. 13], their proof requires only a combined power-type estimate on DF, as in (A_2), rather than the condition (H_3) of [3]). Then, by Theorem 4.1, there exists a sequence $(y_n)_{n\geq 1} \subset \mathbb{R}^N$ such that

$$v_n(\cdot + y_n) \to u \text{ in } H^1$$

for some $u \in H^1$. Then, $(u, \omega) \in M_C$ is a minimiser of E over M_C.

5 Stability results

Lemma 5.1 *Let ϕ be a $H^1(\mathbb{R}^N, \mathbb{R}^k)$ function. Then $|\phi|$ is $H^1(\mathbb{R}^N)$ and*

$$\|D\phi\|_{L^2} \geq \|D|\phi|\|_{L^2}. \tag{7.1}$$

Suppose that for every bounded subset $S \subset \mathbb{R}^N$ $\operatorname{ess\,inf}_S |\phi| > 0$. If equality holds between the two above norms, then there exists λ in $\mathbb{R}^k$ such that $|\lambda| = 1$ and

$$\phi(x) = \lambda|\phi(x)|. \tag{7.2}$$

Proof. The proof of the fact that $|\phi|$ is $H^1(\mathbb{R}^N, \mathbb{R}^k)$ follows the same steps of the case $k = 2$ in [18, Theorem 6.17, p. 152]. Then

$$\partial_{x_i}|\phi| = \begin{cases} \dfrac{\langle \phi, \partial_{x_i}\phi \rangle}{|\phi|} & \text{if } \phi \neq 0 \\ 0 & \text{if } \phi = 0 \end{cases}$$

for every $1 \leq i \leq N$. By the Schwarz inequality,

$$|D|\phi||^2 = \sum_{i=1}^N |\partial_{x_i}|\phi||^2 = \frac{1}{|\phi|^2} \sum_{i=1}^N |\langle \phi, \partial_{x_i}\phi \rangle|^2 \leq \sum_{i=1}^N |\partial_{x_i}\phi|^2 = |D\phi|^2 \tag{7.3}$$

if $\phi \neq 0$. On the region $\{\phi = 0\}$, the same inequality follows easily. Then $D|\phi|$ is L^2. By integrating (7.3), we prove the first part of the statement. Now, we suppose that in (7.1) the equality holds and $|\phi|$ is essentially bounded from below on every bounded subset of $\mathbb{R}^N$. From (7.3) we obtain

$$|\phi||\partial_{x_i}\phi| = |\langle \phi, \partial_{x_i}\phi \rangle|.$$

Because $\phi(x) \neq 0$ a.e., there exists $\mu_i : \mathbb{R}^N \to \mathbb{R}$ such that

$$\partial_{x_i}\phi = \mu_i \phi \text{ a.e.} \tag{7.4}$$

We claim that each of the functions

$$\Lambda_j : \mathbb{R}^N \to \mathbb{R}, \quad x \mapsto \frac{\phi_j(x)}{|\phi(x)|}$$

is constant. From the same approximation argument as [18, Theorem 6.16, p. 178], it follows that Λ_j is $H^1_{loc}(\mathbb{R}^N)$ and

$$|\phi|^3\partial_{x_i}\Lambda_j = \partial_{x_i}\phi_j|\phi|^2 - \phi_j\langle\phi, \partial_{x_i}\phi\rangle = \sum_{h=1}^{k}\partial_{x_i}\phi_j\phi_h^2 - \phi_j\phi_h\partial_{x_i}\phi_h = \mu_i\sum_{h=1}^{k}\phi_j\phi_h^2 - \phi_j\phi_h^2 = 0.$$

The last equality follows from (7.4). So, there exists λ_j in $\mathbb{R}$ with $\Lambda_j \equiv \lambda_j$ a.e. which satisfies (7.2).

A similar result has been proved in [18, Theorem 7.8] in the case $k = 2$, under the assumption that one of the components of ϕ is positive almost everywhere.

Let C be such that $C_j \neq 0$ for $j = 1, 2$. For every (ϕ, ϕ_t) in X such that $\phi_j \neq 0$, we define the map

$$X \ni (\phi, \phi_t) \mapsto \mathbf{P}(\phi, \phi_t) := \left(|\phi_1|, |\phi_2|, \frac{C_1}{\|\phi_1\|^2_{L^2}}, \frac{C_2}{\|\phi_2\|^2_{L^2}}\right) \in M_C. \tag{7.5}$$

Proposition 5.1 *For every $\Phi := (\phi, \phi_t)$ such that $\phi_j \neq 0$, for $j = 1, 2$, there holds*

$$\mathbf{E}(\Phi) \geq E(\mathbf{P}(\Phi)), \quad \mathbf{C}_j(\Phi) = C_j(\mathbf{P}(\Phi)).$$

In the proposition $\mathbf{E}$ and $\mathbf{C}_j$ are the energy and charges defined in (1.6) and (1.7).

Proof. From the Schwarz inequality, we obtain

$$\frac{|\mathbf{C}_j(\phi, \phi_t)|}{\|\phi_j\|_{L^2}} \leq \|\phi_t^j\|_{L^2}. \tag{7.6}$$

By Lemma 5.1 and (7.6),

$$\begin{aligned}\mathbf{E}(\phi, \phi_t) &= \frac{1}{2}\int_{\mathbb{R}^N} |D\phi|^2 + |\phi_t|^2 + 2V(\phi)\\ &\geq \frac{1}{2}\int_{\mathbb{R}^N} |D|\phi||^2 + 2V(|\phi_1|, |\phi_2|) + \frac{1}{2}\sum_{j=1}^{2}\frac{\mathbf{C}_j(\phi, \phi_t)^2}{\|\phi_j\|^2_{L^2}}\\ &= E(\mathbf{P}(\Phi)),\end{aligned}$$

and

$$C_j(\mathbf{P}(\Phi)) = \frac{\mathbf{C}_j(\Phi)}{\|\phi_j\|^2_{L^2}}\int_{\mathbb{R}^N} |\phi_j|^2 = \mathbf{C}_j(\Phi).$$

Proof of Theorem 1.2. Given Φ in Γ_C there exists (u, ω) in K_C and (λ, y) in $\mathbb{T}^2 \times \mathbb{R}^N$ such that

$$\Phi = (\lambda \cdot u(\cdot + y), -i\omega \cdot \lambda \cdot u(\cdot + y)).$$

We used the notation introduced in (1.2). Then

$$\mathbf{E}(\Phi) = E(u, \omega) = m_C, \quad \mathbf{C}_j(\Phi) = \omega_j\|u_j\|^2_{L^2} = C_j.$$

Because $\mathbf{E}$ and $\mathbf{C}_j$ are continuous, if $d(\Phi_n, \Gamma_C) \to 0$, then

$$\mathbf{E}(\Phi_n) \to m_C, \quad \mathbf{C}_j(\Phi_n) \to C_j. \tag{7.7}$$

We prove the converse and suppose that (7.7) holds. We set

$$\Phi_n := (\phi_n, \phi_n^t).$$

Because $C_j \neq 0$ for $j = 1, 2$, $\phi_n^j \not\equiv 0$ for n large enough. Then, it makes sense to define

$$(u_n, \omega_n) := \mathbf{P}(\Phi_n). \tag{7.8}$$

From Proposition 5.1 and (7.7), $(u_n, \omega_n)_{n\geq 1}$ is a minimising sequence of E over M_C. By Theorem 1.1, there are

$$(u, \omega) \in K_C,\ (y_n)_{n\geq 1} \subset \mathbb{R}^N$$

such that

$$u_n = u(\cdot + y_n) + o(1), \quad \omega_n = \omega + o(1). \tag{7.9}$$

We set

$$\psi_n := \phi_n(\cdot - y_n), \quad \psi_n^t := \phi_n^t(\cdot - y_n).$$

By a change of variable, we have

$$\mathbf{E}(\psi_n, \psi_n^t) = \mathbf{E}(\phi_n, \phi_n^t), \quad \mathbf{C}_j(\psi_n, \psi_n^t) = \mathbf{C}_j(\phi_n, \phi_n^t). \tag{7.10}$$

Up to extract a subsequence, we can suppose that there exists (ψ, ψ_t) in X such that

$$\psi_n \rightharpoonup \psi \text{ in } H^1(\mathbb{R}^N, \mathbb{C}^2), \quad \psi_n^t \rightharpoonup \psi_t \text{ in } L^2(\mathbb{R}^N, \mathbb{C}^2). \tag{7.11}$$

By the weak lower semi-continuity of the norm, the strong convergence of $|\psi_n|$, (7.6) and Lemma 5.1, we have

$$\begin{aligned}
\mathbf{E}(\psi_n, \psi_n^t) &= \frac{1}{2}\int_{\mathbb{R}^N} |D\psi_n|^2 + |\psi_n^t|^2 + 2V(\psi_n) \\
&\geq \frac{1}{2}\int_{\mathbb{R}^N} |D\psi|^2 + |\psi_t|^2 + 2V(\psi) \\
&\geq \frac{1}{2}\int_{\mathbb{R}^N} |D|\psi||^2 + 2V(|\psi_1|, |\psi_2|) + \frac{1}{2}\sum_{j=1}^{2} \frac{\mathbf{C}_j(\psi, \psi_t)^2}{\|\psi_j\|_{L^2}^2} \geq m_C.
\end{aligned}$$

Taking the limit as $n \to \infty$, by (7.10), the first of (7.7) and the first of the above inequalities, we obtain

$$\lim_{n\to\infty} \|\psi_n^t\|_{L^2} = \|\psi_t\|_{L^2}, \quad \lim_{n\to\infty} \|D\psi_n\|_{L^2} = \|D\psi\|_{L^2}. \tag{7.12}$$

From the second inequality, we obtain

$$\int_{\mathbb{R}^N} |D\psi_j|^2 = \int_{\mathbb{R}^N} |D|\psi_j||^2, \quad \frac{\mathbf{C}_j(\psi, \psi_t)}{\|\psi_j\|_{L^2}} = \|\psi_t^j\|_{L^2}. \tag{7.13}$$

The weak limit in (7.11) and the strong convergence of $|\psi_n^j|$ to u_j implies that

$$|\psi_j| = u_j \text{ a.e.} \tag{7.14}$$

and

$$\psi_n^j \to \psi_j \text{ in } L^2(\mathbb{R}^N, \mathbb{C}). \tag{7.15}$$

Because (u, ω) is a minimiser of E over M_C, u_j are regular, by (ii), and positive, by (iii) of Proposition 2.2 and (7.14). Thus, ψ_j fulfils the hypotheses of Lemma 5.1. From (7.13) there are λ_j in $\mathbb{C}$ such that $|\lambda_j| = 1$ and

$$\psi_j = \lambda_j|\psi_j| = \lambda_j u_j.$$

The second limit in (7.12) and the first in (7.11) yield

$$D\psi_n^j \to D\psi_j.$$

By (7.15),

$$\psi_n^j \to \lambda_j u_j \text{ in } H^1(\mathbb{R}^N, \mathbb{C}). \tag{7.16}$$

The second equality in (7.13) can be written as

$$\mathrm{Re} \int_{\mathbb{R}^N} \overline{-i\psi_j} \cdot \psi_t^j = \|\psi_t^j\|_{L^2} \|\psi_j\|_{L^2}.$$

Thus, we have an equality between the scalar product and the product of norms. Then

$$\psi_t^j = -i \frac{C_j}{\|\psi_j\|_{L^2}^2} \psi_j. \tag{7.17}$$

From (7.5) and (7.8), we have

$$\omega_n^j = \frac{C_j}{\|\phi_n^j\|_{L^2}^2}.$$

Taking the limit, we obtain

$$\omega_j = \frac{C_j}{\|\psi_j\|_{L^2}^2}.$$

Then (7.17) can be written as

$$\psi_j^t = -i\omega_j \psi_j.$$

By the second limit in (7.11) and the first limit of (7.12)

$$\psi_n^{t,j} \to \psi_t^j = -i\omega_j \lambda_j u_j \text{ in } L^2(\mathbb{R}^N, \mathbb{C}). \tag{7.18}$$

Thus, (7.16) and (7.18) yield

$$d((\psi_n, \psi_n^t), \Gamma_C) \to 0$$

so that $d((\phi_n, \phi_n^t), \Gamma_C) \to 0$.

Proof of Theorem 1.3. The proof of the stability of Γ_C follows from the fact that $\mathbf{V}$, defined in (1.9), is a Lyapunov function (see [3, Definition 2.4]) and from the definition of orbital stability. We prove that $\Gamma(u, \omega)$ is stable if condition (D) is satisfied. We argue by contradiction and suppose that there exists $\varepsilon_0 > 0$ and $(t_n, \Phi_n)_{n \geq 1}$ such that

$$d(\Phi_n, \Gamma(u, \omega)) \to 0, \quad d(U(t_n, \Phi_n), \Gamma(u, \omega)) \geq \varepsilon_0.$$

Thus, there exists (u', ω') in K_C such that

$$\Gamma(u', \omega') \neq \Gamma(u, \omega)$$

and

$$d(U(t_n, \Phi_n), \Gamma(u', \omega')) \to 0 \tag{7.19}$$

By Theorem 1.2, $\mathbf{E}(U(t_n, \Phi_n)) \to m_C$ and

$$(\mathbf{P}(U(t_n, \Phi_n)))_{n \geq 1}$$

is a minimising sequence of E over M_C. By Theorem 1.1, up to extract a subsequence,

$$\mathbf{P}(U(t_n, \Phi_n)) \to G(u'', \omega''). \tag{7.20}$$

for some (u'', ω'') in K_C. By (7.19),

$$\Gamma(u'', \omega'') = \Gamma(u', \omega').$$

Now we set

$$E_\delta := \inf_{\partial B_\delta} E > m_C.$$

The inequality follows from Theorem 1.1 and condition (D). For n large enough,

$$\mathbf{E}(U(t,\Phi_n)) = \mathbf{E}(\Phi_n) < E_\delta$$

for every $t \in \mathbb{R}$. By Proposition 5.1,

$$E_\delta > E(\mathbf{P}(U(t,\Phi_n))).$$

Our assumption on the regularity of the solutions of (CNLKG), ensures that $U(\cdot,\Phi_n)$ is continuous in H^1. Then,

$$\mathbf{P}(U(t_n,\Phi_n)) \in B_\delta(G(u,\omega))$$

otherwise the path $\mathbf{P}(U(\cdot,\Phi_n))$ intersects the boundary of B where $E \geq E_\delta$. By (7.20), $G(u',\omega') \cap B_\delta \neq \emptyset$, so contradicting (D).

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