1. Lines in $\mathbb{R}^2$

For $v, w \in E^2$, we define

$$v \times w := v_1 w_2 - v_2 w_1 \in E.$$

We also define

$$v^\perp := (v_2, -v_1).$$

We have the following:

Proposition 1. *Given $v, w \in E^2$, there holds*

$$v \times w = 0 \Leftrightarrow v \text{ is parallel to } w$$

and $v \cdot v^\perp = 0$.

Proof. Suppose that $v \times w = 0$. Then

$$v_1 w_2 = v_2 w_1$$

Then

$$w_2 v = (w_2 v_1, w_2 v_2) = (v_2 w_1, w_2 v_2) = v_2 (w_1, w_2) = v_2 w.$$

Conversely, suppose that $w \parallel v$. If $w = 0$, then $v \times w = 0$. Otherwise, there exists λ such that

$$v = \lambda w.$$

Then

$$v \times w = (\lambda w) \times w = \lambda (w \times w) = \lambda (w_1 w_2 - w_2 w_1) = 0.$$

As for the second equality, we have

$$v \cdot v^\perp = (v_1, v_2) \cdot (v_2, -v_1) = v_1 v_2 - v_1 v_2 = 0.$$

□

Definition 1 (Parametric form)**.** Given $P \in \mathbb{R}^2$ and $v \in E^2$, a line is the subset of

$$\ell(P, v) := \{P + tv \mid t \in \mathbb{R}\}.$$

If $v = 0$, then $\ell(P, v) = \{P\}$ it is just a point. A point is a *degenerate* line.
The following equalities hold

$$\ell(P, v) = \ell(P, \lambda v) \ \forall \lambda \in \mathbb{R} - \{0\} \tag{1}$$

$$\ell(P, v) = \ell(P + \mu v, v) \ \forall \mu \in \mathbb{R}. \tag{2}$$

In view of the above equalities, the representation of a line with a pair (P, v) is not unique. We wish to state a precise relation between two pairs (P, v) and (Q, w) such that

$$\ell(P, v) = \ell(Q, w).$$

Proposition 2. *Given (P, v) and (Q, w) such that $v, w \neq 0$ there holds*

$$\ell(P, v) = \ell(Q, w) \Leftrightarrow \overrightarrow{PQ} \times v = v \times w = 0.$$

If condition $v = w = 0$, then the proposition fails: just take $P \neq Q$.

Proof. We use the notation

$$\ell := \ell(P, v), \quad \ell' := \ell(Q, w).$$

Firstly, we consider the case $P \neq Q$. If $\ell = \ell'$, then $\ell \subseteq \ell'$. Thus,

$$P \in \ell \Rightarrow P \in \ell'.$$

Therefore, there exists t such that

$$P = Q + tw$$

whence

$$\overrightarrow{QP} = tw \Rightarrow 0 = \overrightarrow{QP} \times w = -\overrightarrow{PQ} \times w.$$

Similarly, from $Q \in \ell$ we obtain

$$\overrightarrow{PQ} \times v = 0.$$

Now, we prove the converse. Suppose that there are two points P, Q and vectors v, w such that

$$\overrightarrow{PQ} \times v = v \times w = 0.$$

Since $v \times w = 0$ and each of the two vectors is non-zero, there exists $\lambda \in \mathbb{R} - \{0\}$ such that

$$w = \lambda v$$

and $\mu \in \mathbb{R}$ such that

$$\overrightarrow{PQ} = \mu v.$$

Then by (1) and (2), we have

$$\ell(Q, w) = \ell(P + \mu v, \lambda v) = \ell(P, v).$$

□

Along with the parametric form, there is a definition of line using cartesian coordinates.

Proposition 3. *Given two points Q, R such that $Q \neq R$, there exists a unique line ℓ such that*

$$Q, R \in \ell.$$

Proof. Firstly, we show that

$$Q, R \in \ell(Q, \overrightarrow{QR}).$$

In fact,

$$Q = Q + 0 \cdot \overrightarrow{QR} \Rightarrow Q \in \ell$$

and

$$R = Q + 1 \cdot \overrightarrow{QR} = Q + (R - Q) = R \Rightarrow R \in \ell.$$

Now, we show that the $\ell(Q, \overrightarrow{QR})$ is the unique line which contains Q and Q. Let $\ell := \ell(P, v)$ be such that $Q \neq R \in \ell(P, v)$. Since $Q, R \in \ell$, there are t_1, t_2 such that

$$Q = P + t_1 v, \quad R = P + t_2 v.$$

Since $Q \neq R$, we have $t_1 \neq t_2$. Then

$$v = \lambda \overrightarrow{QR}, \quad \lambda := \frac{1}{t_2 - t_1} \neq 0.$$

From (1) and (2), there holds

$$\ell(P, v) = \ell(Q - t_1 v, \lambda \overrightarrow{QR}) = \ell(Q, \overrightarrow{QR}).$$

□

Proposition 4 (Intersection of two lines)**.** *Given two lines $\ell := \ell(P, v)$ and $\ell' := \ell(Q, w)$ such that $v, w \neq 0$ and $\ell \neq \ell'$, then*

$$\ell \cap \ell' \neq \emptyset \Leftrightarrow v \times w \neq 0.$$

If $\ell \cap \ell' \neq \emptyset$, then the intersection contains the unique point

$$P + \left(\frac{v^{\perp} \cdot \overrightarrow{PQ}}{v \times w} \right) v.$$

Proof. We argue by contradiction. Suppose that $R \in \ell \cap \ell'$ and $v \times w = 0$. Then, there exists λ such that

$$v = \lambda w, \quad R = Q + tw, \quad R = P + sv.$$

Then, by (2) and (1)

$$\ell(P, v) = \ell(R - sv, v) = \ell(R - s\lambda w, \lambda w) = \ell(R, w) = \ell(Q + tw, w) = \ell(Q, w).$$

We obtained a contradiction with the assumption $\ell \neq \ell'$.

Now, suppose that $v \times w \neq 0$. We prove that

$$\ell \cap \ell' \neq \emptyset.$$

Then, we have to show that there exists a solution to the system

$$P + tv = Q + sw.$$

We write the system coordinate-wise

$$\begin{cases} tv_1 - sw_1 = x_2 - x_1 \\ tv_2 - sw_2 = y_2 - y_1 \end{cases}$$

We multiply the first equation by v_2, the second equation by v_1 and take the difference

$$s(w_1 v_2 - w_2 v_1) = v_1(y_2 - y_1) - v_2(x_2 - x_1).$$

The equation above can be written as

$$s(v \times w) = v^{\perp} \cdot \overrightarrow{PQ}.$$

Then

$$s = \frac{v^{\perp} \cdot \overrightarrow{PQ}}{v \times w}$$

and the intersection point is

$$Q + \left(\frac{v^{\perp} \cdot \overrightarrow{PQ}}{v \times w} \right) w = Q - \left(\frac{v \times \overrightarrow{PQ}}{v \times w} \right) w. \tag{3}$$

□

Definition 2 (Distance between a point and a line)**.** Given a point Q and a line ℓ, we define

$$d(Q, \ell) := \inf\{d(Q, R) \mid R \in \ell\}.$$

Proposition 5. *Given a non-degenerate line $\ell(P, v)$ and a point Q, there holds*

$$d(P, \ell) = \frac{\|v \times \overrightarrow{PQ}\|}{\|v\|}.$$

Proof. We consider the line $\ell' := \ell(Q, v^{\perp})$. By Proposition 4,

$$\ell \cap \ell' \neq \emptyset$$

and, by the second equality in (3), the intersection contains only the point

$$Q' := Q - \left(\frac{v \times \overrightarrow{PQ}}{v \times v^{\perp}} \right) v^{\perp}.$$

Since

$$\overrightarrow{Q'R} \cdot \overrightarrow{Q'Q} = 0$$

for every $R \in \ell$, there holds

$$d(R, Q)^2 = d(R, Q')^2 + d(Q, Q')^2.$$

Then, for every R

$$d(R, Q) \geq d(Q, Q')$$

and the equality holds when $R = Q'$. Thus,

$$d(Q, \ell) = d(Q, Q') = \left\| \left(\frac{v \times \overrightarrow{PQ}}{v \times v^{\perp}} \right) v^{\perp} \right\| = \frac{\|v \times \overrightarrow{PQ}\|}{\|v\|}.$$

$\square$