SOLUTIONS OF THE EXERCISES WEEK ELEVEN

Exercise 1. Suppose that a model satisfies ***A1***, ***A2***, ***A3***, ***A4*** and that there exists a set $x \in \mathscr{U}$. Then, there exists an element y such that $x \neq y$.

Proof. Actually, the exercise need ***A3*** as well.

$$\boldsymbol{A2} \Rightarrow \exists \emptyset$$

Moreover,

$$\emptyset \subseteq x.$$

Then

$$\boldsymbol{A4} \Rightarrow \emptyset \text{ is a set.}$$
$$\boldsymbol{A2} \Rightarrow \exists \{\emptyset\}.$$
$$\boldsymbol{A3} \Rightarrow \{\emptyset\} \text{ is an element.}$$

Then, we have two elements $\emptyset, \{\emptyset\}$.

Cleary, the two elements above are different from each other because the first is empty and the second has at least one element. Then, at least one of the two is different from x. □

Exercise 2. Let A be a partially ordered class. That is, there exists a subclass

$$G \subseteq A \times A$$

such that G is

(Reflexive) $$id_A \subseteq G$$

(Antisymmetric) $$G \cap G^{-1} \subseteq id_A$$

(Transitive) $$G \circ G \subseteq G.$$

Suppose that $\langle A, G \rangle$ is a fully ordered class. Can you express such definition in terms of G?

Proof. If $\langle A, G \rangle$ is a fully ordered class, then

$$\forall x, y \in A \left(x \leq y \vee y \leq x \right)$$

which means

$$\forall (x, y) \in A \times A \left((x, y) \in G \vee (y, x) \in G \right).$$

By definition of inverse graph, we can write

$$\forall (x, y) \in A \times A \left((x, y) \in G \vee (x, y) \in G^{-1} \right)$$

which is equivalent to

$$A \times A \subseteq G \cup G^{-1}.$$

□

Exercise 3. Let $(A, \leq)$ and $(B, \leq)$ two partially ordered class. Let $g \colon A \to B$ be an order-preserving function. Prove the following.

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a) if g is strictly increasing, then for every $a \in A$ there holds

$$\bar{g}(S_a) \subseteq S_{g(a)};$$

b) if A is a **fully-ordered class** and g is strictly increasing and surjective, then for every $a \in A$ there holds

$$\bar{g}(S_a) = S_{g(a)}.$$

Proof.

(a) if $y \in \bar{g}(S_a)$, then there exists $x \in S_a$ such that

$$g(x) = y, \quad x < a.$$

Since g is strictly increasing,

$$y < g(a)$$

which implies

$$y \in S_{g(a)}.$$

Hence

$$\bar{g}(S_a) \subseteq S_{g(a)}.$$

(b) We only need to prove that

$$S_{g(a)} \subseteq \bar{g}(S_a)$$

because we proved the other inclusion on (a). If

$$y \in S_{g(a)}$$

then

$$y < g(a).$$

Since g is surjective, there exists $x \in A$ such that

(1) $$g(x) = y < g(a).$$

Since A is a fully ordered class, either $x < a$ or $a \leq x$. If $a \leq x$, then $g(a) \leq g(x)$, which gives a contradiction with (1). Then

$$x < a$$

whence $y \in \bar{g}(S_a)$.

□