EXERCISES FROM THE TEXT BOOK ("SET THEORY", CHARLES PINTER)

EXERCISES 1.6

Exercise 11 (check [Pin71, ex. 11, page 45]). Prove that $\cap(\mathscr{A} \cup \mathscr{B}) = (\cap\mathscr{A}) \cap (\cup\mathscr{B})$.

Solution.

$$x \in \cap(\mathscr{A} \cup \mathscr{B}) \tag{1}$$

$$(1) \Rightarrow \forall y \in \mathscr{A} \cup \mathscr{B}\, (x \in y) \tag{2}$$

$$(2) \Rightarrow \forall y \in \mathscr{A}\ (x \in y) \tag{3}$$

then

$$x \in \cap\mathscr{A} \tag{4}$$

$$(2) \Rightarrow \forall y \in \mathscr{B}\, (x \in y) \tag{5}$$

then

$$x \in \cap\mathscr{B}. \tag{6}$$

$$(4) \wedge (6) \Rightarrow x \in (\cap\mathscr{A}) \cap (\cap\mathscr{B}) \tag{7}$$

Then

$$x \in (\cap\mathscr{A}) \cap (\cap\mathscr{B}) \tag{8}$$

and

$$\cap(\mathscr{A} \cup \mathscr{B}) \subseteq (\cap\mathscr{A}) \cap (\cup\mathscr{B}).$$

Conversely, suppose that

$$x \in (\cap\mathscr{A}) \cap (\cup\mathscr{B}).$$

Then, (5) and (6) hold. Then

$$\forall y \in \mathscr{A}\ (x \in y)$$

and

$$\forall y \in \mathscr{B}\, (x \in y)$$

whence

$$\forall y \in \mathscr{A} \cup \mathscr{B}\, (x \in y)$$

and

$$x \in \cap(\mathscr{A} \cup \mathscr{B}).$$

□

Exercise 12 (check [Pin71, ex. 12, page 45]). Prove each of the following

a) If $A \in \mathscr{B}$, then $A \subseteq \cup\mathscr{B}$ and $\cap\mathscr{B} \subseteq A$
b) $\mathscr{A} \subseteq \mathscr{B}$ if and only if $\cup\mathscr{A} \subseteq \cup\mathscr{B}$ (the implication $\Leftarrow$ is false, check the counterexample in the solution)
c) If $\varnothing \in \mathscr{A}$, then $\cap\mathscr{A} = \varnothing$

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Solution. a). We prove that $A \subseteq \cup\mathscr{B}$.

Let $x \in A$. By the definition of union, $x \in \cup\mathscr{B}$ if and only if there exists a class $y \in \mathscr{B}$ such that $x \in y$. This class is A.

We prove that

$$\cap\mathscr{B} \subseteq \mathscr{A}.$$

Let $x \in \cap\mathscr{B}$; then, for every $z \in \mathscr{B}$ there holds $x \in z$. In particular, if $z = A$, we obtain $x \in A$.

b). We prove

$$\mathscr{A} \subseteq \mathscr{B} \Rightarrow \cup\mathscr{A} \subseteq \cup\mathscr{B}.$$

$$x \in \cup\mathscr{A} \tag{9}$$

$$x \in \cup\mathscr{A} \Rightarrow \exists y \in \mathscr{A}(x \in y). \tag{10}$$

$$y \in \mathscr{A} \Rightarrow y \in \mathscr{B} \tag{11}$$

$$(x \in y) \wedge (y \in \mathscr{B}) \Rightarrow x \in \cup\mathscr{B}. \tag{12}$$

The implication $\cup\mathscr{A} \subseteq \cup\mathscr{B} \Rightarrow \mathscr{A} \subseteq \mathscr{B}$ is false: if there are at least three elements and Axioms ***A*2, 3** holds, we can set

$$\mathscr{A} := \{\{x\},\{y,z\}\}, \quad \mathscr{B} := \{\{x,y\},\{z\}\}.$$

Clearly,

$$\cup\mathscr{A} = \{x\} \cup \{y,z\} = \{x,y,z\}$$

and

$$\cup\mathscr{B} = \{x,y\} \cup \{z\} = \{x,y,z\}.$$

However

$$\mathscr{A} \cap \mathscr{B} = \emptyset.$$

c). If $\cap\mathscr{A} \neq \emptyset$, there exists $x \in \cap\mathscr{A}$. Then

$$\forall A \in \mathscr{A}(x \in A).$$

In particular, if $A = \emptyset$, we obtain $x \in \emptyset$ which contradicts the definition of $\emptyset$. □

REFERENCES

Pin71. Charles C. Pinter. *Set theory*. Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1971.