1. Triple products

Given $v, w, z \in E^3$, we can consider the product

$$(v \times w) \times z \in E^3.$$

The product above has a simple expression in terms of the vectors v and w. That is,

$$(v \times w) \times z = w(v \cdot z) - v(w \cdot z). \tag{1}$$

Before proving the equality above, we need the following premise. Given a vector $a \in E^3$, we can define the following linear map

$$R(a) = (a_3, a_1, a_2).$$

The operator above is a permutation of the coordinates which is well-behaved with respect to the scalar product and the cross product, as the next proposition shows.

Proposition 1. *Given two vectors a and b, there hold*

$$R(a) \cdot R(b) = a \cdot b \tag{2}$$

$$R(a) \times R(b) = R(a \times b) \tag{3}$$

$$R(a)_1 = a_3. \tag{4}$$

Proof. The equality (4) follows from the definition of R.

$$\begin{aligned} R(a) \cdot R(b) &= (a_3, a_1, a_2) \cdot (b_3, b_1, b_2) = a_3 b_3 + a_1 b_1 + a_2 b_2 \\ &= a_1 b_1 + a_2 b_2 + a_3 b_3 = a \cdot b. \end{aligned}$$

As for (3), we have

$$\begin{aligned} (R(a) \times R(b))_1 &= R(a)_2 R(b)_3 - R(a)_3 R(b)_2 \\ &= a_1 b_2 - a_2 b_1 = (a \times b)_3 = R(a \times b)_1. \\ (R(a) \times R(b))_2 &= R(a)_3 R(b)_1 - R(a)_1 R(b)_3 \\ &= a_2 b_3 - a_3 b_2 = (a \times b)_1 = R(a \times b)_2. \\ (R(a) \times R(b))_3 &= R(a)_1 R(b)_2 - R(a)_2 R(b)_1 \\ &= a_3 b_1 - a_1 b_3 = (a \times b)_2 = R(a \times b)_3. \end{aligned}$$

□

We are now ready to prove the following proposition:

Proposition 2. *Given $v, w, z \in E^3$ there holds*

$$(v \times w) \times z = w(v \cdot z) - v(w \cdot z).$$

Proof. Firstly, we show that the first components of the vectors in (1) are equal. In fact,

$$\begin{aligned} \left((v \times w) \times z\right)_1 &= (v \times w)_2 z_3 - (v \times w)_3 z_2 = (v_3 w_1 - v_1 w_3) z_3 - (v_1 w_2 - v_2 w_1) z_2 \\ &= w_1 (v_3 z_3 + v_2 z_2) - v_1 (w_3 z_3 + w_2 z_2) \\ &= w_1 (v_3 z_3 + v_2 z_2) - v_1 (w_3 z_3 + w_2 z_2) + v_1 w_1 z_1 - v_1 w_1 z_1 \\ &= w_1 (v_3 z_3 + v_2 z_2 + v_1 z_1) - v_1 (w_3 z_3 + w_2 z_2 + w_1 z_1) \\ &= w_1 (v \cdot z) - v_1 (w \cdot z). \end{aligned}$$

Then, given vectors v, w and z, we have

$$\left((v \times w) \times z\right)_1 = w_1 (v \cdot z) - v_1 (w \cdot z). \tag{5}$$

Now, we apply equality (5) to $R(v), R(w)$ and $R(z)$. Then

(6) $$\big((R(v) \times R(w)) \times R(z)\big)_1 = R(w)_1(R(v) \cdot R(z)) - R(v)_1(R(w) \cdot R(z)).$$

By applying (3) two times and (4), it follows that the left term is equal to

(7) $$\big(R(v \times w) \times R(z)\big)_1 = \big(R((v \times w) \times z)\big)_1 = \big((v \times w) \times z\big)_3.$$

By applying (2) to the right term of (6), we obtain

(8) $$R(w)_1(R(v) \cdot R(z)) - R(v)_1(R(w) \cdot R(z)) = w_3(v \cdot z) - v_3(w \cdot z).$$

Then

(9) $$\big((v \times w) \times z\big)_3 = w_3(v \cdot z) - v_3(w \cdot z).$$

We proved that components 1 and 3 of the vectors in (1) are equal. In order to prove that the second component is equal, we consider the operator

$$T(a) := R^2(a) = R(a_3, a_1, a_2) = (a_2, a_3, a_1)$$

defined for every $a \in E^3$. From (2) it follows

(10) $$T(a) \cdot T(b) = a \cdot b.$$

From (3), there holds.

(11) $$\begin{aligned} T(a) \times T(b) &= R^2(a) \times R^2(b) = R(R(a) \times R(b)) \\ &= R^2(a \times b) = T(a \times b). \end{aligned}$$

Moreover,

(12) $$T(a)_1 = a_2$$

for every $a, b \in E^3$. We apply (5) to the vectors $T(v), T(w)$ and $T(z)$. Then

$$\big((T(v) \times T(w)) \times T(z)\big)_1 = T(w)_1(T(v) \cdot T(z)) - T(v)_1(T(w) \cdot T(z)).$$

By (11) and (12) the left member of the equality above equals

$$\big(T((v \times w) \times z)\big)_1 = ((v \times w) \times z)_2$$

By (10), the right member of equals

$$w_2(v \cdot z) - v_2(w \cdot z).$$

Then

(13) $$((v \times w) \times z)_2 = w_2(v \cdot z) - v_2(w \cdot z).$$

Thus, (5), (9) and (13) allows us to conclude the proof. □