Surreal numbers, derivations and transseries

Vincenzo Mantova

Joint work with A. Berarducci

Scuola Normale Superiore, Pisa

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Outline

Conway's games

A **Game** is a pair $L|R$ where L, R are (w.f.) sets of Games and

1. L are the legal moves for **Left** (called **left options**);
2. R are the legal moves for **Right** (called **right options**).

Go, chess, checkers can be interpreted as Games.[1]

Conway defined a partial order and a sum on Games.

He then noticed that some games behave as *numbers*.
I will omit the details of "numbers as Games" and directly jump to a more concrete description.

[1] Ignoring draws, at least!

Surreal numbers as strings

A **surreal number** $x \in \mathbf{No}$ is a string of $+, -$ of ordinal length.

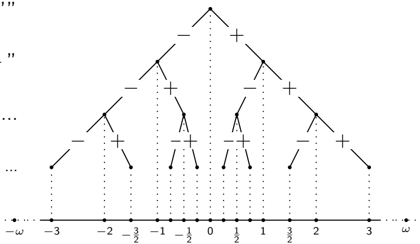
$0 :=$ ""

$1 :=$ " $+$ ", $-1 :=$ " $-$ "

$2 :=$ " $++$ ", $\frac{1}{2} :=$ " $+-$ ", ...

..., $\frac{3}{2} :=$ " $++-$ ", $\frac{3}{4} :=$ " $+--$ ", ...

order on the surreal line:



Go on for all ordinals: $\omega :=$ " $+++\dots$ ", $\frac{1}{\omega} :=$ " $+--\dots$ ", ...
We get $\mathbf{On} \subset \mathbf{No}$, with $\alpha = \underbrace{+++\dots}_{\alpha \text{ times}}$.

Definition. x is **simpler** than y, or $x <_s y$, if x is a prefix of y.

Sum

Definition. Given L, R sets of numbers such that $L < R$, we say $x = L|R$ when x is *simplest* such that $L < x < R$.

For instance: $1 = \{0\}|\{\}$, $2 = \{0, 1\}|\{\}$, $\frac{1}{2} = \{0\}|\{1\}$...
Note. For any $x \in \mathbf{No}$, we can write $x = L|R$ where $L \cup R$ is the set of the numbers strictly simpler than x.

Definition. If $x = \{x'\}|\{x''\}$, $y = \{y'\}|\{y''\}$, then their **sum** is

$$x + y := \{x' + y, x + y'\}|\{x'' + y, x + y''\}.$$

(Idea: we want $(x' + y) < (x + y) < (x'' + y)$...)

Fact. $(\mathbf{No}, +, <)$ is an ordered abelian group.
$(\mathbf{On}, +)$ is a monoid and $+$ is the Hessenberg sum.

Product

(The sum: $x + y = \{x' + y, x + y'\} | \{x'' + y, x + y''\}$.)

Definition. If $x = \{x'\}|\{x''\}$, $y = \{y'\}|\{y''\}$, their **product** is

$$x \cdot y := \{x'y + xy' - x'y', x''y + xy'' - x''y''\} | \\ | \{x'y + xy'' - x'y'', x''y + xy' - x''y'\}.$$

(Idea: we want $(x - x')(y - y') > 0$, $(x - x'')(y - y'') > 0$...)

Fact. $(\mathbf{No}, +, \cdot, <)$ is a field containing $\mathbb{R}$.
$(\mathbf{On}^{>0}, \cdot)$ is a monoid and $\cdot$ is the Hessenberg product.

No as field of Hahn series

Take some $R \supseteq \mathbb{R}$ and consider the **Archimedean** equivalence

$$x \asymp y \leftrightarrow \frac{1}{n}|y| \leq |x| \leq n|y| \text{ for some } n \in \mathbb{N}^{>0}.$$

Let $\Gamma < (R^{>0}, \cdot)$ be a group of representatives for $\asymp$.
Let $\mathbb{R}((\Gamma))$ be the field of **Hahn series**

$$r_0\gamma_0 + r_1\gamma_1 + \dots + r_\omega\gamma_\omega + \dots$$

where $r_\alpha \in \mathbb{R}, \gamma_\alpha \in \Gamma$, and $(\gamma_\alpha)_{\alpha<\gamma}$ decreasing.

Theorem (Hahn-Kaplansky). R embeds into $\mathbb{R}((\Gamma))$.

The **monomials** $\mathfrak{M}$ are the "simplest $\asymp$-representatives" in $\mathbf{No}^{>0}$.
Theorem (Conway). $(\mathfrak{M}, \cdot) \cong (\mathbf{No}, +)$ and $\mathbf{No} \cong \mathbb{R}((\mathfrak{M}))$.
Corollary. **No** is a real closed field (in fact, **Set**-saturated).

Exponentiation

Definition (Kruskal-Gonshor). Given $x = \{x'\} \mid \{x''\}$, define

$$\exp(x) := \left\{0,\ \exp(x')\cdot[x-x']_n,\ \exp(x'')[x-x'']_{2n+1}\right\} \mid \mid \left\{\frac{\exp(x'')}{[x''-x]_n},\ \frac{\exp(x')}{[x'-x]_{2n+1}}\right\},$$

where n ranges in $\mathbb{N}$, $[y]_n := 1 + \frac{y}{1!} + \cdots + \frac{y^n}{n!}$, and $[y]_{2n+1}$ is to be considered only when $[y]_{2n+1} > 0$.

Theorem (Gonshor). exp is a monotone isomorphism $\exp : (\mathbf{No}, +) \tilde{\to} (\mathbf{No}^{>0}, \cdot)$ and $\exp(x) \geq 1 + x$.

Monster model for $\mathbb{R}_{\mathrm{an,exp}}$

Suppose f analytic at $r \in \mathbb{R}$ with $f(r+x) = a_0 + a_1 x + a_2 x^2 + \dots$.
If ε is infinitesimal, we define (after Alling)

$$f(r+\varepsilon) := a_0 + a_1\varepsilon + a_2\varepsilon^2 + \dots.$$

Theorem (van den Dries-Erlich).
$(\mathbf{No}, +, \cdot, <, \{f\}_{f \text{ analytic}}, \exp)$ is an elem. extension of $\mathbb{R}_{\mathrm{an,exp}}$.

By o-minimality and saturation, **No** is a *monster model*.

Hardy fields

Take a family $\mathcal{F}$ of continuous functions $f : (u, \infty) \to \mathbb{R}$.
Take the ring $H(\mathcal{F})$ of **germs at ∞**: for each $f \in \mathcal{F}$,

$$[f] = \{g \in \mathcal{F} \mid g(x) = f(x) \text{ for all } x \text{ sufficiently large}\} .$$

Definition (Bourbaki)**.** $H(\mathcal{F})$ is a **Hardy field** if:

1. it is a field;
2. it is closed under differentiation.

Fact. A Hardy field $H(\mathcal{F})$ is always ordered (given $f \in \mathcal{F}$, either $f(x) > 0$, $f(x) < 0$ or $f(x) = 0$ for all x sufficiently large).

Examples of Hardy fields

Some Hardy fields:

1. (germs of) rational functions $H(\mathbb{R}(x))$;
2. rational functions, exp and log $H(\mathbb{R}(x, \exp(x), \log(x)))$;
3. Hardy's field L of "logarithmico-exponential functions".

Given an expansion R of $\mathbb{R}$, we abbreviate with $H(R)$ the ring of germs at ∞ of unary definable functions $\mathbb{R} \to \mathbb{R}$.

Fact. R is o-minimal if and only if $H(R)$ is a Hardy field.

$H(\mathbb{R}_{\mathrm{an},\exp})$ is a Hardy field which is also an elem. ext. of $\mathbb{R}_{\mathrm{an},\exp}$.

H-fields

H-fields are an abstract version of Hardy fields.
For simplicity, we work over $\mathbb{R}$.

Definition (Aschenbrenner-van den Dries). An ***H*-field** is an ordered field with a derivation D such that:

1. if $x > \mathbb{R}$, then $D(x) > 0$;
2. $D(x) = 0$ if and only if $x \in \mathbb{R}$.

Hardy fields are obviously H-fields.

$H(\mathbb{R}_{\mathrm{an,exp}})$ is an elem. ext. of $\mathbb{R}_{\mathrm{an,exp}}$ which is also an H-field.
It satisfies $D(\exp(f)) = \exp(f)D(f)$, $D(\arctan(f)) = \frac{D(f)}{1+f^2}$, ...

Transseries

$H(\mathbb{R}_{\mathrm{an,exp}})$ is an ordered field: it embeds into some $\mathbb{R}((\Gamma))$.

The field $\mathbb{R}((\Gamma))$ contains series such as

$$\log(\Gamma(t^{-1})) = \log(t) - \gamma t^{-1} + \sum_{n=2}^{\infty} q_n t^{-n}.$$

This is a typical "transseries".

There are many notions of "field of transseries":

1. transseries by Dahn, Göring, Écalle;
2. "*LE*-series" by van den Dries, Macintyre and Marker;
3. "*EL*-(trans)series" by S. Kuhlmann, and Matusinski;
4. "grid-based transseries" by van der Hoeven;
5. "transseries" by M. Schmeling.

Several notions of transseries

The various fields are slightly different from one another.
For instance, *LE* embeds into *EL*, but *EL* contains also:

$$\log(x) + \log(\log(x)) + \log(\log(\log(x)))\ldots.$$

All of them are naturally models of the theory of $\mathbb{R}_{\mathrm{an,exp}}$.
They can be made into *H*-fields (with $D(x) = 1$ for (1)-(3)) such that $D(\exp(t)) = \exp(t)D(t)$, $D(\arctan(t)) = \frac{D(t)}{1+t^2}$, ...

Conj./Theorem (Aschenbrenner-van den Dries-van der Hoeven).
LE-series are a model-companion of *H*-fields.

Surreal numbers as H-fields and transseries?

Theorem (Kuhlmann-Kuhlmann-Shelah). If Γ is a *set*, $\mathbb{R}((\Gamma))$ cannot have a global exp "compatible with the series structure". (We can close under either exp or "infinite sum", but not both).

But **No** is a class, and $\mathbf{No} = \mathbb{R}((\mathfrak{M}))$ has a global exp.

Questions (Aschenbrenner, van den Dries, van der Hoeven, S. Kuhlmann, Matusinski...).

1. Can we give **No** a natural structure of H-field and such that $D(\exp(x)) = \exp(x)D(x)$, $D(\arctan(x)) = \frac{D(x)}{1+x^2}$, ...?
2. Can we give **No** a natural structure of transseries?

Van der Hoeven hinted at a candidate for (2).
S. Kuhlmann and Matusinski made a conjecture for (1)-(2).

Surreal derivations

Definition. A **surreal derivation** is a $D : \mathbf{No} \to \mathbf{No}$ such that:

1. Leibniz' rule: $D(xy) = xD(y) + yD(x)$;
2. strong additivity: $D\left(\sum_{i\in I} a_i\right) = \sum_{i\in I} D(a_i)$;
3. compatibility with exp: $D(\exp(x)) = \exp(x)D(x)$;
4. constant field $\mathbb{R}$: $\ker(D) = \mathbb{R}$;
5. H-field: if $x > \mathbb{R}$ then $D(x) > 0$.

Let us try to construct D and see what happens...

Ressayre representation

Let $\mathbb{J}$ be the **ring of purely infinite numbers** "$\mathbb{R}[[\mathfrak{M}^{>1}]]$".

Theorem (Gonshor). $\exp(\mathbb{J}) = \mathfrak{M}$.

Since **No** $= \mathbb{R}((\mathfrak{M}))$, for any $x \in$ **No** we can write

$$x = r_0 e^{\gamma_0} + r_1 e^{\gamma_1} + \cdots + r_\omega e^{\gamma_\omega} + \ldots$$

where $r_\alpha \in \mathbb{R}$ and $\gamma_\alpha \in \mathbb{J}$, with $(\gamma_\alpha)_{\alpha<\gamma}$ decreasing.

We call this the **Ressayre representation** of x.

Log-atomic numbers

First attempt using the Ressayre representation:

$$D(x) = D\left(r_0 e^{\gamma_0} + r_1 e^{\gamma_1} + \dots\right) = r_0 e^{\gamma_0} D(\gamma_0) + r_1 e^{\gamma_1} D(\gamma_1) + \dots$$

However, this is not inductive, in a very strong sense.

An $x \in \mathbf{No}$ is **log-atomic** if $\mathfrak{m}_0 := x \in \mathfrak{M}$ and
$\mathfrak{m}_{i+1} := \log(\mathfrak{m}_i) \in \mathfrak{M}$ for all $i \in \mathbb{N}$.
Let $\mathbb{L}$ be their class ($\omega \in \mathbb{L}$, $\varepsilon_0 \in \mathbb{L}$, $\kappa_{\mathbf{No}} \subseteq \mathbb{L}$...).

The formula is not informative if $x = \mathfrak{m}_0$ is log-atomic:

$$D(\mathfrak{m}_0) = \mathfrak{m}_0 \cdot D(\mathfrak{m}_1) = \mathfrak{m}_0 \cdot \mathfrak{m}_1 \cdot D(\mathfrak{m}_2) = \dots =?$$

And log-atomic numbers are rather frequent:
Proposition (Berarducci-M.). $\mathbb{L}$ is the "class of levels" of $\mathbf{No}$.

The simplest pre-derivation $\partial_{\mathbb{L}}$

Start with a $D_{\mathbb{L}} : \mathbb{L} \to \mathbf{No}^{>0}$. Axioms (1)-(5) imply

$$\log(D_{\mathbb{L}}(\lambda)) - \log(D_{\mathbb{L}}(\mu)) < \frac{1}{k} \max\{\lambda, \mu\} \text{ for all } \lambda, \mu \in \mathbb{L}, k \in \mathbb{N}^{>0}.$$

Call **pre-derivation** a function satisfying the above inequality.

Proposition (Berarducci-M.). The "simplest" pre-derivation is

$$\partial_{\mathbb{L}}(\lambda) := \exp\left(- \sum_{\substack{\alpha \in \mathbf{On} \\ \exists n : \exp_n(\kappa_{-\alpha}) > \lambda}} \sum_{i=1}^{\infty} \log_i(\kappa_{-\alpha}) + \sum_{i=1}^{\infty} \log_i(\lambda) \right).$$

"Simplest" refers to the simplicity relation $\leq_s$.
$\kappa_{-\alpha}$ are the κ-numbers of S. Kuhlmann and Matusinski.

Ranks

Let us make inductive the formula

$$D(x) = D\left(r_0 e^{\gamma_0} + r_1 e^{\gamma_1} + \dots\right) = r_0 e^{\gamma_0} D(\gamma_0) + r_1 e^{\gamma_1} D(\gamma_1) + \dots$$

Proposition (Berarducci-M.). No $R : \mathbf{No} \to \mathbf{On}$ satisfies

1. $R(x) = 0$ if $x \in \mathbb{L} \cup \mathbb{R}$;
2. otherwise, $R(x) = R(\sum_\gamma r_\gamma e^\gamma) > R(\gamma)$ for $r_\gamma \neq 0$.

Theorem (Berarducci-M.). There is $R : \mathbf{No} \to \mathbf{On}$ such that

1. $R(x) = 0$ if $x \in \mathbb{L} \cup \mathbb{R}$;
2. $R(x) = R(\sum_\gamma r_\gamma e^\gamma) \geq R(\gamma)$ for $r_\gamma \neq 0$, and if the equality holds then γ is *minimal* such that $r_\gamma \neq 0$ (and $r_\gamma = \pm 1$).

Extending $\partial_{\mathbb{L}}$ to $\partial : \mathbf{No} \to \mathbf{No}$

1. if $x \in \mathbb{L}$, $\partial(x) := \partial_{\mathbb{L}}(x)$; if $x \in \mathbb{R}$, $\partial(x) := 0$.
2. $\partial_0(x) := \sum_{R(\gamma)<R(x)} r_\gamma e^\gamma \partial(\gamma)$.
3. if there is a (unique!) γ such that $r_\gamma \neq 0$ and $R(\gamma) = R(x)$,
   1. $\Delta_0(x) := r_\gamma e^\gamma \partial_0(\gamma)$,
   2. $\Delta_{n+1}(x) := r_\gamma e^\gamma \Delta_n(\gamma)$.

   otherwise $\Delta_n(x) := 0$.
4. $\partial(x) := \partial_0(x) + \sum_n \Delta_n(x)$.

Using the inequalities of $\partial_{\mathbb{L}}$ and the properties of R:
Theorem (Berarducci-M.). ∂ is a surreal derivation.

Proposition. ∂ sends infinitesimals to infinitesimals.
Using Rosenlicht "asymptotic integration" and Fodor's lemma:
Theorem. ∂ is surjective (every number has an anti-derivative).

No as a field of transseries

In the PhD thesis of Schmeling:

T4. For all sequences $\mathfrak{m}_i \in \mathfrak{M}$, with $i \in \mathbb{N}$, such that

$$\mathfrak{m}_i = \exp(\gamma_{i+1} + r_{i+1}\mathfrak{m}_{i+1} + \delta_{i+1})$$

we have eventually $r_{i+1} = \pm 1$ and $\delta_{i+1} = 0$.

Theorem (Berarducci-M.). **No** satisfies T4, and therefore it is a field of transseries as defined by Schmeling.
This is roughly van der Hoeven's conjecture.

Theorem (Fornasiero). Every model of the theory of $\mathbb{R}_{\mathrm{an,exp}}$ embeds "initially" in **No** (hence the image is truncation-closed).
Therefore, the models have a structure of (Schmeling) transseries.

Open questions

1. Complete van der Hoeven's picture.
2. Relationship with *LE*, *EL*, ...
3. Differential equations solved in $(\mathbf{No}, \partial)$?
4. Pfaffian functions?
5. Elementary extension of *LE*?
6. Transexponential functions?
7. ...

Thanks for your attention