ON TANGENT CONES TO LENGTH MINIMIZERS IN CARNOT–CARATHÉODORY SPACES

ROBERTO MONTI, ALESSANDRO PIGATI, AND DAVIDE VITTONE

Abstract. We give a detailed proof of some facts about the blow-up of horizontal curves in Carnot–Carathéodory spaces.

1. Introduction

One of the main open problems in sub-Riemannian geometry regards the study of the regularity of length minimizing curves. Several related technical issues appear when one uses a blow-up procedure to pass to the *nilpotent approximation* around a given point. Typical problems one has to face when passing to such a "tangent" structure concern: proving that any blow-up κ of a length minimizer γ is length minimizing in the nilpotent approximation; proving that κ is parametrized by arclength if so is γ; proving that a curve with left (respectively, right) derivative gives a (suitably defined) left (resp., right) half-line κ in the blow-up.

In this paper we give detailed proofs of these facts: in a special case, some of them were already sketched in [17, Section 3.2]. Though of a technical nature, these results are crucially used in [9, 11, 14]. The papers [9, 11] deal with the minimality problem for curves with corner-type singularities, i.e., for curves possessing a point where left and right derivatives to the curve exist and are not equal. In [14] it was shown that the *tangent cone* (see Definition 3.1) to a length minimizer at any of its (interior) points contains a horizontal line; in doing so, one uses the rich algebraic structure of the nilpotent approximation and, actually, of the Carnot group lifting the nilpotent approximation, see Section 4 below.

Let M be a connected n-dimensional C^∞-smooth manifold and $\mathscr{X} = \{X_1, \dots, X_r\}$, $r \geq 2$, a system of C^∞-smooth vector fields on M that are pointwise linearly independent and satisfy the Hörmander condition introduced below. We call the pair $(M, \mathscr{X})$ a *Carnot–Carathéodory (CC) structure*. Given an interval $I \subseteq \mathbb{R}$, a Lipschitz curve

2010 *Mathematics Subject Classification.* 53C17, 49K30, 28A75.

Key words and phrases. Length minimizers, Carnot–Carathéodory spaces, sub-Riemannian geometry, Carnot groups.

R. M. and D. V. are supported by MIUR (Italy) and University of Padova STARS Project SUGGESTION ("Sub-Riemannian Geometry and Geometric Measure Theory Issues: Old and New"). D. V. is also supported by University of Padova Project Networking and INdAM-GNAMPA Project 2017 "Campi vettoriali, superfici e perimetri in geometrie singolari".

$\gamma : I \to M$ is said to be *horizontal* if there exist functions $h_1, \dots, h_r \in L^\infty(I)$ such that for a.e. $t \in I$ we have

$$\dot\gamma(t) = \sum_{i=1}^r h_i(t) X_i(\gamma(t)). \tag{1.1}$$

The function $h \in L^\infty(I;\mathbb{R}^r)$ is called the *control* of γ. Letting $|h| := (h_1^2 + \ldots + h_r^2)^{1/2}$, the length of γ is then defined as

$$L(\gamma) := \int_I |h(t)|\, dt.$$

Since M is connected, by the Chow–Rashevsky theorem (see e.g. [4, 16, 3]) for any pair of points $x, y \in M$ there exists a horizontal curve joining x to y. We can therefore define a distance function $d : M \times M \to [0,\infty)$ letting

$$d(x,y) := \inf \big\{ L(\gamma) \mid \gamma : [0,T] \to M \text{ horizontal with } \gamma(0) = x \text{ and } \gamma(T) = y \big\}. \tag{1.2}$$

The resulting metric space (M,d) is a *Carnot–Carathéodory space*. Since our analysis is local, our results apply in particular to *sub-Riemannian manifolds* $(M, \mathscr{D}, g)$, where $\mathscr{D} \subset TM$ is a completely non-integrable distribution and g is a smooth metric on $\mathscr{D}$.

If the closure of any ball in (M,d) is compact, then the infimum in (1.2) is a minimum, i.e., any pair of points can be connected by a length-minimizing curve. A horizontal curve $\gamma : [0,T] \to M$ is a *length minimizer* if $L(\gamma) = d(\gamma(0), \gamma(T))$.

The main contents of the paper are the following:

(i) we define a tangent Carnot–Carathéodory structure $(M^\infty, \mathscr{X}^\infty)$ at any point of M, using exponential coordinates of the first kind, see Section 2;

(ii) in Section 3, we define the tangent cone for a horizontal curve, at a given time, as the set of all possible blow-ups in $(M^\infty, \mathscr{X}^\infty)$ of the curve, and we show that this cone is always nonempty, see Proposition 3.2;

(iii) we show that, if the curve has a right derivative at the given time, the (positive) tangent cone consists of a single half-line, see Theorem 3.5;

(iv) if the curve is a length minimizer, in Theorem 3.6 we show that all the blow-ups are length minimizers in $(M^\infty, \mathscr{X}^\infty)$, as well;

(v) in Section 4, we show that a tangent Carnot–Carathéodory structure can be lifted to a free Carnot group. Most of the results in this section are well known (see [10]). However, we add some details on the stability of length minimality under lifting.

In this paper we chose to work in exponential coordinates of the first kind. Some of the results hold and are well known in a general system of privileged coordinates. However, one of the key results, namely Theorem 3.5, is valid only in exponential coordinates of the first kind: see Remark 3.10. In Remark 3.11 we discuss a statement of Theorem 3.5 valid in a general system of privileged coordinates. Also, it is possible to define the tangent cone in a coordinate-free way based solely on the controls: see Remark 3.13.

We also decided to work with a system of pointwise linearly independent vector fields (constant rank). This assumption makes the proof of Theorem 2.3 less complicated. We believe that our results also hold for a system of Hörmander vector fields with varying rank.

Finally, it is our pleasure to thank the two anonymous referees. Their careful reading and detailed comments helped us to improve the paper in a substantial way.

2. Nilpotent approximation: definition of a tangent structure

In this section we introduce some basic notions about Carnot–Carathéodory spaces. Then we describe the structure of a specific frame of vector fields $Y_1, \dots, Y_n$ (constructed below) in exponential coordinates, see Theorem 2.3. We also prove a lemma describing the infinitesimal behaviour of the Carnot–Carathéodory distance d near 0, with respect to suitable anisotropic dilations, see Lemma 2.4. General references for this section are [1, 2, 10].

We denote by $\mathrm{Lie}(X_1, \dots, X_r)$ the real Lie algebra generated by $X_1, \dots, X_r$ through iterated commutators. The evaluation of this Lie algebra at a point $x \in M$ is a vector subspace of the tangent space T_xM. If, for any $x \in M$, we have

$$\mathrm{Lie}(X_1, \dots, X_r)(x) = T_xM,$$

we say that the system $\mathscr{X} = \{X_1, \dots, X_r\}$ satisfies the *Hörmander condition* and we call the pair $(M, \mathscr{X})$ a *Carnot–Carathéodory (CC) structure*.

Given a point $x_0 \in M$, let $\varphi \in C^\infty(U; \mathbb{R}^n)$ be a chart such that U is an open neighborhood of x_0 and $\varphi(x_0) = 0$. Then $V := \varphi(U)$ is an open neighborhood of $0 \in \mathbb{R}^n$ and the system of vector fields $Y_i := \varphi_* X_i$, with $i = 1, \dots, r$, still satisfies the Hörmander condition in V.

For a multi-index $J = (j_1, \dots, j_k)$ with $k \geq 1$ and $j_1, \dots, j_k \in \{1, \dots, r\}$, define the iterated commutator

$$Y_J := [Y_{j_1}, \dots, Y_{j_{k-1}}, Y_{j_k}]$$

where, here and in the following, for given vector fields $V_1, \dots, V_q$ we use the short notation $[V_1, \dots, V_q]$ to denote the commutator $[V_1, [\cdots, [V_{q-1}, V_q] \cdots]]$. We say that Y_J is a commutator of *length* $\ell(J) := k$ and we denote by L^j the linear span of $\{Y_J(0) \mid \ell(J) \leq j\}$, so that

$$\{0\} = L^0 \subseteq L^1 \subseteq \cdots \subseteq L^s = \mathbb{R}^n$$

for some minimal $s \geq 1$. We select multi-indices $J_1 = (1), \dots, J_r = (r), J_{r+1}, \dots, J_n$ such that, for each $1 \leq j \leq s$,

$$\ell(J_{\dim L^{(j-1)}+1}) = \cdots = \ell(J_{\dim L^j}) = j$$

and such that, setting $Y_i := Y_{J_i}$, the vectors $Y_1(0), \dots, Y_{\dim L^j}(0)$ form a basis of L^j. In particular, we have $\dim L^1 = r$.

Possibly composing φ with a diffeomorphism (and shrinking U and V), we can assume that for any point $x = (x_1, \dots, x_n) \in V$ we have

$$x = \exp\Big(\sum_{i=1}^n x_i Y_i\Big)(0). \tag{2.3}$$

Such coordinates $(x_1, \dots, x_n)$ are called *exponential coordinates of the first kind* associated with the frame $Y_1, \dots, Y_n$. To each coordinate x_i we assign the weight $w_i := \ell(J_i)$ and we define the anisotropic dilations $\delta_\lambda : \mathbb{R}^n \to \mathbb{R}^n$

$$\delta_\lambda(x) := (\lambda^{w_1} x_1, \dots, \lambda^{w_n} x_n), \quad \lambda > 0. \tag{2.4}$$

Definition 2.1. A function $f : \mathbb{R}^n \to \mathbb{R}$ is *δ-homogeneous of degree* $w \in \mathbb{N}$ if $f(\delta_\lambda(x)) = \lambda^w f(x)$ for all $x \in \mathbb{R}^n$, $\lambda > 0$. We will refer to such a w as the *δ-degree* of f.

We will frequently use the anisotropic (pseudo-)norm

$$\|x\| := \sum_{i=1}^n |x_i|^{1/w_i}, \quad x \in \mathbb{R}^n. \tag{2.5}$$

The norm function, $x \mapsto \|x\|$, is δ-homogeneous of degree 1.

We recall two facts about the exponential map, which are discussed e.g. in [15, pp. 141–147]. First, for any $\psi \in C^\infty(V)$, we have the Taylor expansion

$$\psi\Big(\exp\Big(\sum_{i=1}^n s_i Y_i\Big)(0)\Big) \sim \left(\mathrm{e}^{\sum_i s_i Y_i}\psi\right)(0) \tag{2.6}$$

where

- the left-hand side is a function of $s \in \mathbb{R}^n$ near 0;
- the right-hand side is a shorthand for the formal series

$$\sum_{k=0}^\infty \frac{1}{k!}((s_1 Y_1 + \dots + s_n Y_n)^k \psi)(0) = \sum_{k=0}^\infty \frac{1}{k!} \sum_{i_1, \dots, i_k \in \{1, \dots, n\}} s_{i_1} \cdots s_{i_k} (Y_{i_1} \cdots Y_{i_k} \psi)(0);$$

- given a smooth function $f(x)$ and a formal power series $S(x)$, we define the relation $f(x) \sim S(x)$ if the formal Taylor series of $f(x)$ at 0 is $S(x)$.

Second, letting $S := \sum_{i=1}^n s_i Y_i$ and $T := \sum_{i=1}^n t_i Y_i$, the following formal Taylor expansions hold as well:

$$\psi\Big(\exp(S) \circ \exp(T)(0)\Big) \sim \left(\mathrm{e}^T \mathrm{e}^S \psi\right)(0) = (\mathrm{e}^{P(T,S)}\psi)(0), \tag{2.7}$$

where

$$P(T,S) := \sum_{p=1}^\infty \frac{(-1)^{p+1}}{p} \sum_{k_i + \ell_i \geq 1} \frac{[T^{k_1}, S^{\ell_1}, \dots, T^{k_p}, S^{\ell_p}]}{k_1! \cdots k_p! \ell_1! \cdots \ell_p! (k_1 + \ell_1 + \dots + k_p + \ell_p)}. \tag{2.8}$$

Above, the notation T^k stands for $T, \dots, T$, k times.

Remark 2.2. The formal power series identity $\mathrm{e}^T \mathrm{e}^S = \mathrm{e}^{P(T,S)}$ is a purely algebraic fact which holds in any (noncommutative, graded, complete) associative real algebra, see e.g. [8, Section X.2]: this principle will be used in the proofs of Theorem 2.3 and Lemma 2.4.

The following theorem is proved using exponential coordinates of the first kind. In the case of exponential coordinates of the *second* kind, the theorem is proved in [7]. Theorem 2.3 is also used in [5], where fine properties of functions with bounded variation (with respect to a family of vector fields) are studied using exponential coordinates of the first kind.

Theorem 2.3. The vector fields $Y_1, \dots, Y_n$ are of the form

$$Y_i(x) = \sum_{j=i}^{n} a_{ij}(x) \frac{\partial}{\partial x_j}, \qquad x \in V, \quad i = 1, \dots, n, \tag{2.9}$$

where $a_{ij} \in C^\infty(V)$ are functions such that $a_{ij} = p_{ij} + r_{ij}$ and:

(i) for $w_j \geq w_i$, p_{ij} are δ-homogeneous polynomials in $\mathbb{R}^n$ of degree $w_j - w_i$;
(ii) for $w_j \leq w_i$, $p_{ij} = \delta_{ij}$ (in particular, $p_{ij} = 0$ for $w_j < w_i$);
(iii) $r_{ij} \in C^\infty(V)$ satisfy $r_{ij}(0) = 0$;
(iv) for $w_j \geq w_i$, $r_{ij}(x) = o(\|x\|^{w_j - w_i})$ as $x \to 0$.

Proof. Suppose for a moment that

$$a_{ij}(x) = O(\|x\|^{w_j - w_i}), \qquad i, j = 1, \dots, n, \ w_j \geq w_i. \tag{2.10}$$

Let p_{ij} be the sum of all monomials of δ-degree $w_j - w_i$ in the Taylor expansion of a_{ij}, with the convention that $p_{ij} = 0$ if $w_j < w_i$. Statements (i) and (iv) then hold by construction, while (ii) and (iii) follow from $a_{ij}(0) = \delta_{ij}$, which is a consequence of (2.3).

Let us show (2.10). We pullback the identity $Y_i(x) = \sum_j a_{ij}(x) \frac{\partial}{\partial x_j}$ to the origin using the map $\exp(-X)$ (locally defined near x), where $X := \sum_k x_k Y_k$, for a fixed $x \in V$. We have

$$\exp(-X)_*(Y_i(x)) = \sum_j a_{ij}(x) \exp(-X)_* \Big(\frac{\partial}{\partial x_j}(x) \Big), \tag{2.11}$$

where the sum ranges from 1 to n. The above equation reads

$$\sum_\ell b_{i\ell}(x) Y_\ell(0) = \sum_{j,\ell} a_{ij}(x) c_{j\ell}(x) Y_\ell(0)$$

for suitable smooth coefficients $b_{i\ell}(x), c_{j\ell}(x)$. We claim that

$$b_{i\ell}(x) = O(\|x\|^{w_\ell - w_i}), \quad c_{j\ell}(x) = O(\|x\|^{w_\ell - w_j}), \quad \text{and} \quad c_{j\ell}(0) = \delta_{j\ell}.$$

Then, defining $A := (a_{ij})$, $B := (b_{i\ell})$ and $C := 1 - (c_{j\ell})$ (1 denoting the identity matrix), we obtain three $n \times n$ matrices satisfying $B(x) = A(x)(1 - C(x))$ and $C(0) = 0$. In particular, $1 - C(x)$ is invertible for x close to 0 and $(1 - C(x))^{-1} = \sum_{p=0}^{\infty} C(x)^p$. This gives

$$A(x) = \sum_{p=0}^{s} B(x)C(x)^p + o(|x|^s) = \sum_{p=0}^{s} B(x)C(x)^p + o(\|x\|^s)$$

for any $s \in \mathbb{N}$, and (2.10) easily follows.

The proof of $c_{j\ell}(0) = \delta_{j\ell}$ follows from the definition of $c_{j\ell}$ and from $\frac{\partial}{\partial x_j} = Y_j(0)$, which in turn comes from (2.3), as already observed.

We prove the claim $b_{i\ell}(x) = O(\|x\|^{w_\ell - w_i})$. By (2.3), the left-hand side of (2.11) satisfies

$$\exp(-X)_*(Y_i(x)) = \frac{d}{dt} \exp(-X) \circ \exp(tY_i) \circ \exp(X)(0)\Big|_{t=0}.$$

Using (2.7) and Remark 2.2, for any smooth ψ we obtain

$$\psi\big(\exp(-X) \circ \exp(tY_i) \circ \exp(X)(0)\big) \sim \mathrm{e}^{P(P(X,tY_i),-X)}\psi(0),$$

the left-hand side being interpreted as a function of (x, t). We now differentiate this identity at $t = 0$. Since $W(t) := P\big(P(X, tY_i), -X\big)$ vanishes at $t = 0$, one has $\frac{d}{dt}(\mathrm{e}^{W(t)}\psi)(0)\Big|_{t=0} = \frac{d}{dt}(W(t)\psi)(0)\Big|_{t=0}$ and, letting ψ range among the coordinate functions, we deduce that any finite-order expansion in x of $\exp(-X)_*(Y_i(x))$ is a linear combination of terms of the form

$$x_{i_1} \cdots x_{i_p} [Y_{i_1}, \ldots, Y_{i_m}, Y_i, Y_{i_{m+1}}, \ldots, Y_{i_p}](0)$$

where $p \geq 1$ and $0 \leq m \leq p$. By Jacobi's identity, the iterated commutator $[Y_{i_1}, \ldots, Y_{i_m}, Y_i, Y_{i_{m+1}}, \ldots, Y_{i_p}](0)$ is a linear combination of the vectors $Y_J(0)$ with $\ell(J) = \overline{w} := \sum_{q=1}^{p} w_{i_q} + w_i$ and so, by construction, it is a linear combination of the vectors $Y_\ell(0)$ with $w_\ell \leq \overline{w}$. Hence, letting $w_\alpha := \sum_{q=1}^{n} \alpha_q w_q$ for all $\alpha \in \mathbb{N}^n$, we have

$$\exp(-X)_*(Y_i(x)) \sim \sum_{\ell} \sum_{\alpha : w_\alpha \geq w_\ell - w_i} d_{\alpha i \ell} x^\alpha Y_\ell(0),$$

for suitable coefficients $d_{\alpha i \ell} \in \mathbb{R}$. This gives the required estimate.

The proof of $c_{j\ell}(x) = O(\|x\|^{w_\ell - w_j})$ is analogous to the preceding argument, once we observe that

$$\exp(-X)_*\Big(\frac{\partial}{\partial x_j}(x)\Big) = \frac{d}{dt} \exp(-X) \circ \exp(X + tY_j)(0)\Big|_{t=0}.$$

We can omit the details. $\square$

Lemma 2.4. For any compact set $K \subset \mathbb{R}^n$ and any $\varepsilon > 0$ there exist $\eta > 0$ and $\overline{\lambda} > 0$ such that $\lambda d(\delta_{1/\lambda}(x), \delta_{1/\lambda}(y)) < \varepsilon$ for all $x, y \in K$ with $|x - y| < \eta$ and all $\lambda \geq \overline{\lambda}$.

Proof. Let $\psi \in C^\infty(V)$ be an arbitrary smooth function. Using (2.6) and Remark 2.2, we have the following identity of formal power series in $(s,t) \in \mathbb{R}^n \times \mathbb{R}^n$: letting $S := \sum_{i=1}^n s_i Y_i$ and $T := \sum_{i=1}^n t_i Y_i$,

$$\psi(\exp(S)(0)) \sim (\mathrm{e}^S \psi)(0) = (\mathrm{e}^T \mathrm{e}^{-T} \mathrm{e}^S \psi)(0) = (\mathrm{e}^T \mathrm{e}^{P(-T,S)} \psi)(0). \tag{2.12}$$

The truncation $P_N(-T,S)$ of the series $P(-T,S)$ up to δ-degree $N := w_n$ is

$$P_N(-T,S) = \sum_{1 \le \ell(J) \le N} q_J(s,t) Y_J, \tag{2.13}$$

where the sum is over all J such that $1 \le \ell(J) \le N$ and q_J is a homogeneous polynomial with δ-degree $\ell(J)$, i.e., $q_J(\delta_\lambda s, \delta_\lambda t) = \lambda^{\ell(J)} q_J(s,t)$. This follows from the fact that any iterated commutator $[Y_{i_1}, \ldots, Y_{i_k}]$ is a constant linear combination of the vector fields Y_J's with $\ell(J) = \sum_{j=1}^k w_{i_j}$ (which in turn is a consequence of Jacobi's identity).

Moreover, using (2.13) and applying (2.7) with the vector fields Y_J in place of $Y_1, \ldots, Y_n$, we have the following formal Taylor expansion in (s,t) at $0 \in \mathbb{R}^{2n}$

$$\psi\big(\exp(P_N(-T,S)) \circ \exp(T)(0)\big) \sim \Big(\mathrm{e}^T \mathrm{e}^{P_N(-T,S)} \psi\Big)(0),$$

which, by (2.12), coincides with the one of $\psi(\exp(S)(0))$ up to δ-degree N. Since this holds for any ψ, we deduce (for instance letting ψ range among the coordinate functions) that

$$\exp(S)(0) = \exp(P_N(-T,S)) \circ \exp(T)(0) + o(|s|^N + |t|^N),$$

which by (2.3) gives

$$s = \exp(P_N(-T,S))(t) + o(|s|^N + |t|^N) =: f(s,t) + o(|s|^N + |t|^N).$$

Now let $s = \delta_{1/\lambda}(x)$ and $t = \delta_{1/\lambda}(y)$ with $x, y \in K$. Since

$$q_J(s,t) = \lambda^{-\ell(J)} q_J(x,y),$$

by [15, Theorem 4] we get

$$d(t, f(s,t)) \le C \sum_{1 \le \ell(J) \le N} |q_J(s,t)|^{1/\ell(J)} = C\lambda^{-1} \sum_{1 \le \ell(J) \le N} |q_J(x,y)|^{1/\ell(J)},$$

while, by [15, Lemma 2.20(b)],

$$d(s, f(s,t)) = O(|s - f(s,t)|^{1/w_n}) = o(|s| + |t|) = o(\lambda^{-1}),$$

provided λ is sufficiently large. Thus, by the triangle inequality,

$$\lambda d(\delta_{1/\lambda}(x), \delta_{1/\lambda}(y)) = \lambda d(s,t) \le C \sum_{1 \le \ell(J) \le N} |q_J(x,y)|^{1/\ell(J)} + \frac{\varepsilon}{2}$$

for all $\lambda \geq \overline{\lambda}$, for a suitably large $\overline{\lambda} > 0$. Finally, since $P_N(S,-S) = 0$, we can assume that q_J vanishes on the diagonal of $K \times K$ (possibly replacing $q_J(s,t)$ with $q_J(s,t) - q_J(s,s)$). Hence, by compactness of K, we also have

$$C \sum_{1 \leq \ell(J) \leq N} |q_J(x,y)|^{1/\ell(J)} < \frac{\varepsilon}{2}$$

whenever $x, y \in K$ are such that $|x-y| < \eta$, for a suitably small $\eta > 0$. □

We now introduce the vector fields $Y_1^\infty, \ldots, Y_r^\infty$ in $\mathbb{R}^n$ defined by

$$Y_i^\infty(x) := \sum_{j=1}^n p_{ij}(x) \frac{\partial}{\partial x_j},$$

and we let $\mathscr{X}^\infty = \{Y_1^\infty, \ldots, Y_r^\infty\}$. The vector fields $Y_1^\infty, \ldots, Y_r^\infty$ are known as the *nilpotent approximation* of $Y_1, \ldots, Y_r$ at the point 0. In the literature, they are sometimes denoted by $\widehat{Y}_i$. By Proposition 2.5 below, the pair $(\mathbb{R}^n, \mathscr{X}^\infty)$ is a Carnot–Carathéodory structure. We set $M^\infty := \mathbb{R}^n$ and we call $(M^\infty, \mathscr{X}^\infty)$ a *tangent* Carnot–Carathéodory structure to $(M, \mathscr{X})$ at the point $x_0 \in M$.

Proposition 2.5. The vector fields $Y_1^\infty, \ldots, Y_r^\infty$ are pointwise linearly independent and satisfy the Hörmander condition in $\mathbb{R}^n$. Moreover, any iterated commutator $Y_J^\infty := [Y_{j_1}^\infty, [\ldots, [Y_{j_{k-1}}^\infty, Y_{j_k}^\infty] \ldots]]$ of length $\ell(J) = k > s$ vanishes identically.

Proof. We claim that Theorem 2.3 implies $Y_i^\infty = \lim_{\lambda\to\infty} \lambda^{-1}(\delta_\lambda)_* Y_i$, for all $i = 1, \ldots, r$, in the (local) C^∞-topology (the vector field $\lambda^{-1}(\delta_\lambda)_* Y_i$ being defined on $\delta_\lambda(V)$). Indeed, since $Y_i(x) = Y_i^\infty(x) + \sum_j r_{ij}(x) \frac{\partial}{\partial x_j}$, we have

$$\lambda^{-1}((\delta_\lambda)_* Y_i)(x) = Y_i^\infty(x) + \sum_{j=1}^n \lambda^{w_j - 1} r_{ij}(\delta_{1/\lambda}(x)) \frac{\partial}{\partial x_j},$$

because $\lambda^{-1}(\delta_\lambda)_* Y_i^\infty = Y_i^\infty$. By Theorem 2.3, the monomials in the Taylor expansion of r_{ij} have δ-degree greater than $w_j - 1$. Thus, for any $\alpha \in \mathbb{N}^n$,

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha}(\lambda^{w_j-1} r_{ij}(\delta_{1/\lambda}(x))) = \lambda^{w_j - 1 - w_\alpha} \frac{\partial^{|\alpha|} r_{ij}}{\partial x^\alpha}(\delta_{1/\lambda}(x)),$$

where $w_\alpha := \sum_\ell \alpha_\ell w_\ell$. The monomials in the expansion of $\frac{\partial^{|\alpha|} r_{ij}}{\partial x^\alpha}$ have δ-degree greater than $w_j - 1 - w_\alpha$, hence $\left|\frac{\partial^{|\alpha|} r_{ij}}{\partial x^\alpha}(\delta_{1/\lambda}(x))\right| = o(\lambda^{-(w_j - 1 - w_\alpha)})$ and the claim follows.

In particular, we deduce that for any multi-index J

$$Y_J^\infty = \lim_{\lambda\to\infty} \lambda^{-\ell(J)}(\delta_\lambda)_* Y_J, \tag{2.14}$$

in the local C^∞-topology. Hence, defining the $n \times n$ matrix $D_\lambda :=\mathrm{diag}[\lambda^{w_1}, \ldots, \lambda^{w_n}]$ and recalling that $\ell(J_p) = w_p$, for all $p = 1, \ldots, n$ we have

$$Y_{J_p}^\infty(x) = \lim_{\lambda\to\infty} \lambda^{-w_p} D_\lambda Y_{J_p}(\delta_{1/\lambda}(x)).$$

Now the first statement follows from

$$\begin{aligned}\det(Y^\infty_{J_1},\dots,Y^\infty_{J_n})(x) &= \lim_{\lambda\to\infty}\lambda^{-\sum_i w_i}\det(D_\lambda)\det(Y_{J_1},\dots,Y_{J_n})(\delta_{1/\lambda}(x))\\ &= \det(Y_{J_1},\dots,Y_{J_n})(0)=\det(Y_1,\dots,Y_n)(0),\end{aligned}$$

which is a nonzero constant. This gives the first part of the statement.

In order to prove the last assertion, we use again the fact that $\lambda^{-1}(\delta_\lambda)_*Y_i^\infty = Y_i^\infty$ for $i=1,\dots,r$. For any $x\in\mathbb{R}^n$ and any J with $\ell(J)>s=w_n$ we have, by (2.14),

$$Y_J^\infty(x)=\lim_{\lambda\to\infty}\lambda^{-\ell(J)}((\delta_\lambda)_*Y_J)(x)=\lim_{\lambda\to\infty}\lambda^{-\ell(J)}D_\lambda Y_J(\delta_{1/\lambda}(x)).$$

The right-hand side is bounded by $\lambda^{s-\ell(J)}|Y_J(\delta_{1/\lambda}(x))|$ (if $\lambda\geq 1$), which tends to 0 as $\lambda\to\infty$. This shows that $Y_J^\infty=0$. □

Remark 2.6. Setting $Y_i^\infty := Y_{J_i}^\infty$ for $i=1,\dots,n$, the coordinate functions on $M^\infty=\mathbb{R}^n$ are exponential coordinates of the first kind for $(Y_1^\infty,\dots,Y_n^\infty)$, namely

$$x=\exp\Big(\sum_{i=1}^n x_iY_i^\infty\Big)(0). \tag{2.15}$$

for any $x\in\mathbb{R}^n$. This follows from the fact that, for λ large enough (depending on x), we have $y:=\delta_{\lambda^{-1}}(x)\in V$ and, using (2.3) with y in place of x,

$$x=\delta_\lambda\Big(\exp\Big(\sum_i y_iY_i\Big)(0)\Big)=\exp\Big(\sum_i x_i\lambda^{-w_i}(\delta_\lambda)_*Y_i\Big)(0)\to\exp\Big(\sum_i x_iY_i^\infty\Big)(0)$$

as $\lambda\to\infty$, since (2.14) gives $\lambda^{-w_i}(\delta_\lambda)_*Y_i\to Y_i^\infty$ in the local C^∞-topology.

3. The tangent cone to a horizontal curve

Let $(M,\mathscr{X})$ be a CC structure and let $\gamma:[-T,T]\to M$ be a horizontal curve. Given $t\in(-T,T)$, let φ be a chart centered at $x_0=\gamma(t)$, as in the previous section, together with the dilations δ_λ and the tangent CC structure $(M^\infty,\mathscr{X}^\infty)$ introduced above.

Definition 3.1. The *tangent cone* $\mathrm{Tan}(\gamma;t)$ to γ at $t\in(-T,T)$ is the set of all horizontal curves $\kappa:\mathbb{R}\to M^\infty$ such that there exists an infinitesimal sequence $\eta_i\downarrow 0$ satisfying, for any $\tau\in\mathbb{R}$,

$$\lim_{i\to\infty}\delta_{1/\eta_i}\varphi\big(\gamma(t+\eta_i\tau)\big)=\kappa(\tau),$$

with uniform convergence on compact subsets of $\mathbb{R}$.

We remark that any limit curve as above is automatically $(M^\infty,\mathscr{X}^\infty)$-horizontal: see e.g. the proof of Theorem 3.6.

The definition of $\mathrm{Tan}(\gamma;t)$ depends on the choice $Y_1,\dots,Y_n$ of linearly independent iterated commutators. It also depends on the chart φ. However, the tangent cone can be described in a coordinate-free way in terms of the controls: see Remark 3.13.

When $\gamma : [0, T] \to M$, the tangent cones $\mathrm{Tan}^+(\gamma; 0)$ and $\mathrm{Tan}^-(\gamma; T)$ can be defined in a similar way: $\mathrm{Tan}^+(\gamma; 0)$ contains curves in M^∞ defined on $[0, \infty)$, while $\mathrm{Tan}^-(\gamma; T)$ contains curves defined on $(-\infty, 0]$.

When $M = M^\infty$ or $M = G$ is a Carnot group, there is already a group of dilations on M itself. In such cases, when $\gamma(t) = 0$, we define the tangent cone $\mathrm{Tan}(\gamma; t)$ as the set of horizontal limit curves of the form $\kappa(t) = \lim_{i\to\infty} \delta_{1/\eta_i}\gamma(t + \eta_i\tau)$.

The tangent cone is closed under uniform convergence of curves on compact sets.

Proposition 3.2. For any horizontal curve $\gamma : [-T, T] \to M$ the tangent cone $\mathrm{Tan}(\gamma; t)$ is nonempty for any $t \in (-T, T)$. The same holds for $\mathrm{Tan}^+(\gamma; 0)$ and $\mathrm{Tan}^-(\gamma; T)$, for a horizontal curve $\gamma : [0, T] \to M$.

Proof. We prove that $\mathrm{Tan}^+(\gamma; 0) \neq \emptyset$. The other cases are analogous.

We use exponential coordinates of the first kind centered at $\gamma(0)$. By (1.1), we have a.e.

$$\dot\gamma = \sum_{i=1}^r h_i Y_i(\gamma) = \sum_{j=1}^n \sum_{i=1}^r h_i a_{ij}(\gamma) \frac{\partial}{\partial x_j},$$

where $h_i \in L^\infty([0, T])$ and $a_{ij} = p_{ij} + r_{ij}$, as in Theorem 2.3. Letting $K := \gamma([0, T])$, we have $|\dot\gamma(t)| \le C$ for some constant depending on $\|a_{ij}\|_{L^\infty(K)}$ and $\|h\|_{L^\infty}$. This implies that $|\gamma(t)| \le Ct$ for all $t \in [0, T]$.

By induction on $k \ge 1$, we prove the following statement: for any j satisfying $w_j \ge k$ we have $|\gamma_j(t)| \le Ct^k$. The base case $k = 1$ has already been treated. Now assume that $w_j \ge k > 1$ and that the statement is true for $1, \dots, k-1$. Since r_{ij} is smooth, we have $r_{ij} = q_{ij,k} + r_{ij,k}$, where $q_{ij,k}$ is a polynomial containing only terms with δ-homogeneous degree at least $w_j - w_i + 1 = w_j$ and $|r_{ij,k}(x)| \le C|x|^{k-1}$ on K (here $|x|$ denotes the usual Euclidean norm).

Each monomial $c_\alpha x^\alpha$ of the polynomial $p_{ij} + q_{ij,k}$ has δ-degree $w_\alpha \ge w_j - 1$. If $\alpha_m = 0$ whenever $w_m \ge k$, then we can estimate

$$|\gamma(t)^\alpha| = \prod_{m: w_m \le k-1} |\gamma_m(t)|^{\alpha_m} \le Ct^{w_\alpha} \le Ct^{k-1},$$

using the inductive hypothesis with k replaced by $w_m \le k-1$. Otherwise, there exists some index m with $w_m \ge k$ and $\alpha_m > 0$, in which case

$$|\gamma(t)^\alpha| \le C|\gamma_m(t)| \le Ct^{k-1},$$

using the inductive hypothesis with k replaced by $k-1$. Thus $|p_{ij}(\gamma(t)) + q_{ij,k}(\gamma(t))| \le Ct^{k-1}$. Combining this with the estimate $|r_{ij,k}(\gamma(t))| \le Ct^{k-1}$, we obtain $|a_{ij}(\gamma(t))| \le Ct^{k-1}$. So we finally have

$$|\gamma_j(t)| \le \|h\|_{L^\infty} \sum_{i=1}^r \int_0^t |a_{ij}(\gamma(\tau))|\, d\tau \le Ct^k,$$

completing the inductive proof. Applying the above statement with $k = w_j$, we obtain

$$|\gamma_j(t)| \le Ct^{w_j}, \tag{3.16}$$

for a suitable constant C depending only on K, T and $\|h\|_{L^\infty}$.

Now we prove that $\mathrm{Tan}^+(\gamma;0)$ is nonempty. For $\eta > 0$ consider the family of curves $\gamma^\eta(t) := \delta_{1/\eta}(\gamma(\eta t))$, defined for $t \in [0, T/\eta]$. The derivative of γ^η is a.e.

$$\dot\gamma^\eta(t) = \sum_{j=1}^{n}\sum_{i=1}^{r} h_i(\eta t)\eta^{1-w_j} a_{ij}(\gamma(\eta t))\frac{\partial}{\partial x_j},$$

where, by Theorem 2.3 and the estimates (3.16), we have

$$|a_{ij}(\gamma(\eta t))| \le C\|\gamma(\eta t)\|^{w_j-1} \le C(\eta t)^{w_j-1}.$$

This proves that the family of curves $(\gamma^\eta)_{\eta>0}$ is locally Lipschitz equicontinuous. By Ascoli–Arzelà theorem and a diagonal argument, there exists a subsequence $(\gamma^{\eta_i})_i$ that is converging locally uniformly as $\eta_i \to 0$ to a curve $\kappa : [0,\infty) \to \mathbb{R}^n$. □

Remark 3.3. The following result was obtained along the proof of Proposition 3.2. Let $(M, \mathscr{X})$ be a Carnot–Carathéodory structure. Using exponential coordinates of the first kind, we (locally) identify M with $\mathbb{R}^n$ and we assign to the coordinate x_j the weight w_j, as above. Given $T > 0$ and K compact, there exists a positive constant $C = C(K, T)$ such that the following holds: for any horizontal curve $\gamma : [0, T] \to K$ parametrized by arclength and such that $\gamma(0) = 0$, one has

$$|\gamma_j(t)| \le Ct^{w_j}, \quad \text{for any } j = 1, \dots, n \text{ and } t \in [0, T]. \tag{3.17}$$

In Carnot groups, by homogeneity, the constant C is independent of K and T.

Definition 3.4. We say that $v \in \mathbb{R}^n$ is a *right tangent vector* to a curve $\gamma : [0, T] \to \mathbb{R}^n$ at 0 if

$$\gamma(t) = tv + o(t), \quad \text{as } t \to 0^+.$$

The definition of a *left tangent vector* is analogous.

The next result is stated in exponential coordinates of the first kind: see Remark 3.11 for a statement which holds in general systems of privileged coordinates. Recall that $V = \varphi(U)$ is the image of the chart.

Theorem 3.5. Let $\gamma : [0, T] \to V$ be a horizontal curve parametrized by arclength, with $\gamma(0) = 0$. If γ has a right tangent vector $v \in \mathbb{R}^n$ at 0, then:

(i) $v_j = 0$ for $j > r$ and $|v| \le 1$;
(ii) $\mathrm{Tan}^+(\gamma; 0) = \{\kappa\}$, where $\kappa(t) = tv$ for $t \in [0, \infty)$;
(iii) $|v| = 1$ if γ is also length minimizing.

A similar statement holds if $\gamma : [-T, 0] \to V$ has a left tangent vector at 0.

Proof. (i) Since $Y_i(x) = \frac{\partial}{\partial x_i} + o(1)$ as $x \to 0$, we have

$$\gamma_j(t) = \int_0^t \sum_{i=1}^r h_i(s)\delta_{ij}\,ds + o(t). \tag{3.18}$$

We deduce that $v_j = 0$ for $j > r$ and

$$|v| = \lim_{t\to 0^+} \left|\frac{\gamma(t)}{t}\right| \le \lim_{t\to 0^+} \frac{1}{t}\int_0^t |h(s)|\,ds = 1.$$

(ii) Since $\gamma_j(t) = v_j t + o(t)$ for $j \le r$, it suffices to show that

$$\gamma_j(t) = o(t^{w_j}), \quad j > r. \tag{3.19}$$

Up to a rotation of the vector fields $Y_1, \dots, Y_r$, which by (2.3) corresponds to a rotation of the first r coordinates, we can assume that $v_2 = \ldots = v_r = 0$. Notice that Theorem 2.3 still applies in these new exponential coordinates. From (3.18) we get

$$\lim_{t\to 0^+} \frac{1}{t}\int_0^t h_i(s)\,ds = \begin{cases} v_1 & i = 1 \\ 0 & i = 2, \dots, r. \end{cases} \tag{3.20}$$

By Remark 3.3 we have $\|\gamma(t)\| = O(t)$. We now show (3.19) by induction on $j \ge r+1$.

Assume the claim holds for $1, \dots, j-1$. The coordinate γ_j, with $j > r$, is

$$\gamma_j(t) = \sum_{i=1}^r \int_0^t h_i(s)a_{ij}(\gamma(s))\,ds = \int_0^t h_1(s)a_{1j}(\gamma(s))\,ds + \sum_{i=2}^r \int_0^t h_i(s)a_{ij}(\gamma(s))\,ds.$$

By Theorem 2.3, $a_{ij} = p_{ij} + r_{ij}$ with $r_{ij}(x) = o(\|x\|^{w_j-1})$, so we deduce that

$$a_{ij}(\gamma(s)) = p_{ij}(\gamma(s)) + r_{ij}(\gamma(s)) = p_{ij}(\gamma(s)) + o(s^{w_j-1}), \quad i = 1, \dots, r.$$

From (2.3) we deduce that for $i = 1, \dots, r$ we have $Y_i(0, \dots, x_i, \dots, 0) = \frac{\partial}{\partial x_i}$, hence

$$a_{ij}(0, \dots, x_i, \dots, 0) = 0, \quad j > r. \tag{3.21}$$

The polynomial $p_{ij}(x)$ is δ-homogeneous of degree $w_j - w_i = w_j - 1$ and so it contains no variable x_k with $k \ge j$. Condition (3.21) implies that $p_{ij}(x)$ does not contain the monomial $x_i^{w_j-1}$, either. Thus, when $i = 1$ each monomial in $p_{1j}(x)$ contains at least one of the variables $x_2, \dots, x_{j-1}$. By the inductive assumption, it follows that $p_{1j}(\gamma(s)) = o(s^{w_j-1})$, and thus $a_{1j}(\gamma(s)) = o(s^{w_j-1})$. This implies that

$$\int_0^t h_1(s)a_{1j}(\gamma(s))\,ds = o(t^{w_j}).$$

Now we consider the case $i = 2, \dots, r$. Letting $p_{ij} = c_{ij}x_1^{w_j-1} + \widehat{p}_{ij}$ with $c_{ij} \in \mathbb{R}$ and $\widehat{a}_{ij} := \widehat{p}_{ij} + r_{ij}$, we have $\widehat{a}_{ij}(\gamma(s)) = o(s^{w_j-1})$ as in the previous case and thus

$$\int_0^t h_i(s)\widehat{a}_{ij}(\gamma(s))\,ds = o(t^{w_j}).$$

We claim that, for $i = 2, \dots, m$, we also have

$$\int_0^t h_i(s)\gamma_1(s)^{w_j-1}\,ds = o(t^{w_j}).$$

Indeed, since $v_i = 0$ we have $H_i(s) := \int_0^s h_i(s')\,ds' = o(s)$, so integration by parts gives

$$\begin{aligned}\int_0^t h_i(s)\gamma_1(s)^{w_j-1}\,ds &= H_i(t)\gamma_1(t)^{w_j-1} - (w_j-1)\int_0^t H_i(s)\gamma_1(s)^{w_j-2}\dot\gamma_1(s)\,ds\\ &= o(t^{w_j}) + \int_0^t o(s^{w_j-1})\,ds = o(t^{w_j}).\end{aligned}$$

This ends the proof of (3.19) and hence of (ii).

(iii) By Theorem 3.6 below, κ is parametrized by arclength. But $(v_1, \dots, v_r)$ equals its (continuous) control $h(t)$ at $t = 0$, so $|v| = 1$. □

For $\lambda > 0$, we define the vector fields $Y_1^\lambda, \dots, Y_r^\lambda$ in $\delta_\lambda(V)$ by

$$Y_i^\lambda(x) := \lambda^{-1}((\delta_\lambda)_* Y_i)(x) = \sum_{j=1}^n \lambda^{w_j-1} a_{ij}(\delta_{1/\lambda}(x))\frac{\partial}{\partial x_j}, \quad x \in \delta_\lambda(V).$$

In the proof of Proposition 2.5 it was shown that

$$Y_i^\lambda \to Y_i^\infty \tag{3.22}$$

locally uniformly in $\mathbb{R}^n$ as $\lambda \to \infty$, together with all the derivatives.

We denote by d^λ the Carnot–Carathéodory metric of $(\delta_\lambda(V), \mathscr{X}^\lambda)$, with $\mathscr{X}^\lambda := \{Y_1^\lambda, \dots, Y_r^\lambda\}$. The distance function d^λ is related to the distance function d via the formula

$$d^\lambda(x,y) = \lambda d\big(\delta_{1/\lambda}(x), \delta_{1/\lambda}(y)\big), \tag{3.23}$$

for all $x, y \in \delta_\lambda(V)$ and $\lambda > 0$. Indeed, let $\gamma : [0,1] \to V$ be a horizontal curve

$$\gamma(t) = \gamma(0) + \int_0^t \sum_{i=1}^r h_i(s) Y_i(\gamma(s))\,ds, \quad t \in [0,1], \tag{3.24}$$

and define the curve $\gamma^\lambda : [0,\lambda] \to \delta_\lambda(V)$

$$\gamma^\lambda(t) := \delta_\lambda \gamma(t/\lambda), \quad t \in [0,\lambda]. \tag{3.25}$$

Then we have

$$\gamma^\lambda(t) = \gamma^\lambda(0) + \int_0^t \sum_{i=1}^r h_i(s/\lambda) Y_i^\lambda(\gamma^\lambda(s))\,ds, \quad t \in [0,\lambda], \tag{3.26}$$

and therefore the length of γ^λ is

$$L^\lambda(\gamma^\lambda) = \int_0^\lambda \big|h(s/\lambda)\big|\,ds = \lambda\int_0^1 |h(s)|\,ds = \lambda L(\gamma). \tag{3.27}$$

If γ is length minimizing, then the curves in $\mathrm{Tan}(\gamma;t)$ are also locally length minimizing. This is the content of the next theorem.

Theorem 3.6. Let $\gamma : [-T,T] \to M$ be a length-minimizing curve in $(M,\mathscr{X})$, parametrized by arclength, and let $\gamma^\infty \in \mathrm{Tan}(\gamma;t_0)$ for some $t_0 \in (-T,T)$. Then γ^∞ is horizontal, parametrized by arclength and, when restricted to any compact interval, it is length minimizing in the tangent Carnot–Carathéodory structure $(M^\infty,\mathscr{X}^\infty)$.

Proof. We can assume $t_0 = 0$. We use exponential coordinates of the first kind centered at $\gamma(0)$. Given any $\overline{T} > 0$, for some sequence $\lambda_j \to \infty$ we have

$$\gamma^{\lambda_j}(t) := \delta_{\lambda_j}\gamma(t/\lambda_j) \to \gamma^\infty(t) \qquad \text{in } L^\infty([-\overline{T},\overline{T}]). \tag{3.28}$$

With abuse of notation, we write $\lambda = \lambda_j$ and we replace $j \to \infty$ with $\lambda \to \infty$. Up to a subsequence, we can assume that the functions $h(t/\lambda)$ weakly converge in $L^2([-\overline{T},\overline{T}];\mathbb{R}^r)$ to some $h^\infty \in L^2([-\overline{T},\overline{T}];\mathbb{R}^r)$ such that $|h^\infty| \le 1$ almost everywhere. Then, using (3.26), we have

$$\gamma^\infty(t) = \lim_{\lambda\to\infty} \int_0^t \sum_{i=1}^r h_i(s/\lambda) Y_i^\lambda(\gamma^\lambda(s))\,ds = \int_0^t \sum_{i=1}^r h_i^\infty Y_i^\infty(\gamma^\infty(s))\,ds,$$

so γ^∞ is $(M^\infty,\mathscr{X}^\infty)$-horizontal and, denoting by d^∞ the Carnot–Carathéodory distance on M^∞ induced by the family $\mathscr{X}^\infty$, its length satisfies

$$d^\infty(\gamma^\infty(-\overline{T}),\gamma^\infty(\overline{T})) \le L^\infty\left(\gamma^\infty\big|_{[-\overline{T},\overline{T}]}\right) = \int_{-\overline{T}}^{\overline{T}} |h^\infty|\,dt \le 2\overline{T}. \tag{3.29}$$

We will see that, in fact, the converse inequality $d^\infty(\gamma^\infty(-\overline{T}),\gamma^\infty(\overline{T})) \ge 2\overline{T}$ holds as well, thus proving that γ^∞ is length minimizing on $[-\overline{T},\overline{T}]$ and parametrized by arclength (with control h^∞).

Let $\kappa^\infty : [-\overline{T},\overline{T}] \to \mathbb{R}^n$ be an $(M^\infty,\mathscr{X}^\infty)$-horizontal curve such that $\kappa^\infty(\pm\overline{T}) = \gamma^\infty(\pm\overline{T})$, with control $k^\infty \in L^\infty([-\overline{T},\overline{T}];\mathbb{R}^n)$. For all λ large enough, the ordinary differential equation

$$\dot\kappa^\lambda(t) = \sum_{i=1}^r k_i^\infty(t) Y_i^\lambda(\kappa^\lambda(t)) \tag{3.30}$$

with initial condition $\kappa^\lambda(-\overline{T}) = \kappa^\infty(-\overline{T})$ has a (unique) solution defined on $[-\overline{T},\overline{T}]$. Indeed, let K be a compact neighborhood of $\kappa^\infty([-\overline{T},\overline{T}])$. For any $\varepsilon > 0$ we have $\|Y_i^\lambda - Y_i^\infty\|_{L^\infty(K)} \le \varepsilon$ eventually. If $-\overline{T} \in I \subseteq [-\overline{T},\overline{T}]$ is the maximal (compact) subinterval such that κ^λ is defined on I and $\kappa^\lambda(I) \subseteq K$, we have

$$|\dot\kappa^\lambda - \dot\kappa^\infty| \le C\varepsilon + C\sum_{i=1}^r |Y_i^\infty(\kappa^\lambda) - Y_i^\infty(\kappa^\infty)| \le C\varepsilon + C|\kappa^\lambda - \kappa^\infty|$$

on I, for some C depending on $\|k^\infty\|_{L^\infty}$ and $\|\nabla Y_i^\infty\|_{L^\infty(K)}$. Hence, by Gronwall's inequality, $|\kappa^\lambda - \kappa^\infty| \leq C\varepsilon$ on I. If ε is small enough, we deduce that $\kappa^\lambda(\max I)$ belongs to the interior of K, so $I = [-\overline{T}, \overline{T}]$. Since ε was arbitrary, we also get

$$\lim_{\lambda\to\infty} \kappa^\lambda(\pm\overline{T}) = \kappa^\infty(\pm\overline{T}) = \gamma^\infty(\pm\overline{T}) = \lim_{\lambda\to\infty} \gamma^\lambda(\pm\overline{T}). \tag{3.31}$$

From the length minimality of γ^λ in $(\delta_\lambda(V), \mathscr{X}^\lambda)$ it follows that

$$\begin{aligned} 2\overline{T} = L^\lambda\Big(\gamma^\lambda\big|_{[-\overline{T},\overline{T}]}\Big) &\leq L^\lambda(\kappa^\lambda) + d^\lambda\Big(\kappa^\lambda(-\overline{T}), \gamma^\lambda(-\overline{T})\Big) + d^\lambda\Big(\kappa^\lambda(\overline{T}), \gamma^\lambda(\overline{T})\Big) \\ &= \int_{-\overline{T}}^{\overline{T}} |k^\infty(t)|\, dt + \lambda d\Big(\delta_{1/\lambda}\kappa^\lambda(-\overline{T}), \delta_{1/\lambda}\gamma^\lambda(-\overline{T})\Big) \\ &\quad + \lambda d\Big(\delta_{1/\lambda}\kappa^\lambda(\overline{T}), \delta_{1/\lambda}\gamma^\lambda(\overline{T})\Big). \end{aligned}$$

By Lemma 2.4 and (3.31), we have

$$\lim_{\lambda\to\infty} \lambda d(\delta_{1/\lambda}\kappa^\lambda(\pm\overline{T}), \delta_{1/\lambda}\gamma^\lambda(\pm\overline{T})) = 0.$$

Hence, $2\overline{T} \leq \int_{-\overline{T}}^{\overline{T}} |k^\infty(t)|\, dt = L^\infty(\kappa^\infty)$. Since κ^∞ was arbitrary, we conclude that $d^\infty(\gamma^\infty(-\overline{T}), \gamma^\infty(\overline{T})) \geq 2\overline{T}$. □

The following fact is a special case of the general principle according to which the tangent to the tangent is (contained in the) tangent.

Proposition 3.7. Let $\gamma : [-T, T] \to M$ be a horizontal curve and $t \in (-T, T)$. If $\kappa \in \operatorname{Tan}(\gamma; t)$ and $\widehat{\kappa} \in \operatorname{Tan}(\kappa; 0)$, then $\widehat{\kappa} \in \operatorname{Tan}(\gamma; t)$.

Proof. We can assume without loss of generality that $t = 0$. We use exponential coordinates of the first kind centered at $\gamma(0)$. Let $N > 0$ be fixed. Since $\widehat{\kappa} \in \operatorname{Tan}(\kappa; 0)$, there exists an infinitesimal sequence $\xi_k \downarrow 0$ such that, for all $t \in [-N, N]$ and $k \in \mathbb{N}$, we have

$$\|\widehat{\kappa}(t) - \delta_{1/\xi_k}\kappa(\xi_k t)\| \leq \frac{1}{2^k}.$$

Since $\kappa \in \operatorname{Tan}(\gamma; 0)$, there exists an infinitesimal sequence $\eta_k \downarrow 0$ such that, for all $t \in [-N, N]$ and $k \in \mathbb{N}$, we have

$$\|\kappa(\xi_k t) - \delta_{1/\eta_k}\gamma(\eta_k \xi_k t)\| \leq \frac{\xi_k}{2^k}.$$

It follows that for the infinitesimal sequence $\sigma_k := \xi_k\eta_k$ we have, for all $t \in [-N, N]$,

$$\|\widehat{\kappa}(t) - \delta_{1/\sigma_k}\kappa(\sigma_k t)\| \leq \|\widehat{\kappa}(t) - \delta_{1/\xi_k}\kappa(\xi_k t)\| + \|\delta_{1/\xi_k}\kappa(\xi_k t) - \delta_{1/\sigma_k}\gamma(\sigma_k t)\| \leq \frac{1}{2^{k-1}}.$$

The thesis now follows by a diagonal argument. □

When $\gamma : [0, T] \to M$, there are analogous versions of Propositions 3.6 and 3.7 for $\operatorname{Tan}^+(\gamma; 0)$ and $\operatorname{Tan}^-(\gamma; T)$.

Proposition 3.8. Let $\kappa : \mathbb{R} \to M^\infty$ be a horizontal curve in $(M^\infty, \mathscr{X}^\infty)$. The following statements are equivalent:

(i) there exist $c_1, \ldots, c_r \in \mathbb{R}$ such that $\dot\kappa = \sum_{i=1}^r c_i Y_i^\infty(\kappa)$ and $\kappa(0) = 0$;

(ii) there exists $x_0 \in M^\infty$ such that $\kappa(t) = \delta_t(x_0)$ (here δ_t is defined by (2.4) also for $t \le 0$).

Proof. We prove (i)⇒(ii). Since $(\delta_\lambda)_* Y_i^\infty = \lambda Y_i^\infty$ for $\lambda \neq 0$, the curve $\delta_\lambda \circ \kappa(\cdot/\lambda)$ satisfies the same differential equation, so $\delta_\lambda \circ \kappa(t/\lambda) = \kappa(t)$; choosing $\lambda = t$ we get $\kappa(t) = \delta_t(\kappa(1))$.

We check (ii)⇒(i). Up to rescaling time, we can assume that $\dot\kappa(1)$ exists and is a linear combination of $Y_1^\infty(\kappa(1)), \ldots, Y_r^\infty(\kappa(1))$, so $\dot\kappa(1) = \sum_i \overline{h}_i Y_i^\infty(\kappa(1))$ for some $\overline{h} \in \mathbb{R}^r$. If h is the control of κ, for a.e. s we have

$$\sum_{i=1}^r \overline{h}_i Y_i^\infty(\kappa(1)) = \dot\kappa(1) = s\frac{d}{dt}\kappa(t/s)\Big|_{t=s} = s\frac{d}{dt}(\delta_{1/s} \circ \kappa(t))\Big|_{t=s} = \sum_{i=1}^r h_i(s) Y_i^\infty(\kappa(1)),$$

again because $s(\delta_{1/s})_* Y_i^\infty = Y_i^\infty$. Since $Y_1^\infty, \ldots, Y_r^\infty$ are pointwise linearly independent (see Proposition 2.5), we get $h = \overline{h}$ a.e. □

Definition 3.9. We say that a horizontal curve κ in $(M^\infty, \mathscr{X}^\infty)$ is a *horizontal line* (through 0) if one of the conditions (i)–(ii) of Proposition 3.8 holds.

The definition of *positive and negative half-line* is similar, the formulas above being required to hold for $t \ge 0$ and $t \le 0$, respectively.

Remark 3.10. One of the referees pointed out to us the following example, for which we thank him once again. In $\mathbb{R}^3$ consider the vector fields $X = \partial_x$ and $Y = \partial_y + x\partial_z$. The coordinates (x, y, z) are privileged and the curve $\gamma(t) = (t, t, t^2/2)$, $t \in \mathbb{R}$, has constant control $h(t) = (1, 1)$ and satisfies (i) and (ii) of Proposition 3.8. So it is a horizontal line in the sense of Definition 3.9, but it does not satisfy property (ii) of Theorem 3.5, nor (3.19): the reason is that the vector fields X and Y are not in exponential coordinates of the first kind.

Remark 3.11. In general systems of privileged coordinates, the fact that $v_j = 0$ for $j > r$ (whenever γ has a right tangent vector v) always holds, since $|\gamma_j(t)| \le Ct^2$ by definition of privileged coordinates. Moreover, in view of Remark 3.13 below, the conclusions of Theorem 3.5 hold once they are replaced with the following more robust statements:

(i') $|w| \le 1$, for the constant control $w \in \mathbb{R}^r$ specified in (ii');

(ii') $\mathrm{Tan}^+(\gamma; 0) = \{\kappa\}$, where the curve κ has constant control $w \in \mathbb{R}^r$, i.e., γ is a horizontal line in the sense of Definition 3.9 (observe that the proof of Proposition 3.8 works in general privileged coordinates);

(iii') $|w| = 1$ if γ is also length minimizing.

Also, $w = (v_1, \dots, v_r)$ if the privileged coordinates satisfy $Y_i(x) = \frac{\partial}{\partial x_i} + o(1)$ for $i = 1, \dots, r$, as is readily seen from (3.18) and the fact that $\int_0^t h_i(s)\,ds = w_i t + o(t)$.

On the other hand, in exponential coordinates of the first kind (i) and (ii) of Proposition 3.8 are equivalent to the fact that κ is a straight horizontal line, i.e., that $\kappa(t) = tv$ for some $v \in \mathbb{R}^n$ such that $v_{r+1} = \cdots = v_n = 0$.

Remark 3.12. Let us observe the following fact. Let $\gamma : [-T, T] \to M$ be a length minimizer parametrized by arclength with control $h = (h_1, \dots, h_r)$ and let $t \in (-T, T)$ be fixed. Then, the tangent cone $\mathrm{Tan}(\gamma; t)$ contains a horizontal line κ in M^∞ if and only if there exist an infinitesimal sequence $\eta_i \downarrow 0$ and a constant unit vector $c \in S^{r-1}$ such that

$$h(t + \eta_i \,\cdot) \to c \qquad \text{in } L^2_{loc}(\mathbb{R}, \mathbb{R}^r).$$

As usual, an analogous version holds for $\mathrm{Tan}^+(\gamma; 0)$ and $\mathrm{Tan}^-(\gamma; T)$ in case γ is a length minimizer parametrized by arclength on the interval $[0, T]$.

Let us prove our claim; we can set $t = 0$. Assume that there exists a sequence $\eta_i \downarrow 0$ such that the curves $\gamma^i(\tau) := \delta_{1/\eta_i}\varphi(\gamma(\eta_i \tau))$ converge locally uniformly to a horizontal line κ in the tangent CC structure $(M^\infty, \mathscr{X}^\infty)$; we have

$$\gamma^i(\tau) = \int_0^\tau \sum_{j=1}^r h_j(\eta_i s) Y_j^{1/\eta_i}(\gamma^i(s))\,ds.$$

Up to subsequences we have $h(\eta_i \,\cdot) \rightharpoonup h_\infty$ in $L^2_{loc}(\mathbb{R}, \mathbb{R}^r)$, with $\|h_\infty\|_{L^\infty} \le 1$. Since $Y_j^{1/\eta_i} \to Y_j^\infty$ locally uniformly, writing $h_\infty = (h_\infty^1, \dots, h_\infty^r)$ we obtain

$$\kappa(\tau) = \int_0^\tau \sum_{j=1}^r h_\infty^j(s) Y_j^\infty(\kappa(s))\,ds.$$

By Proposition 3.6, κ is parametrized by arclength. So $|h_\infty| = 1$ a.e. and, since κ is a horizontal line, h_∞ is constant. Finally, for any compact set $K \subset \mathbb{R}$, we trivially have $\|h(\eta_i \,\cdot)\|_{L^2(K,\mathbb{R}^r)} \to \|h_\infty\|_{L^2(K,\mathbb{R}^r)}$, which gives $h(\eta_i \,\cdot) \to h_\infty$ in $L^2(K, \mathbb{R}^r)$. The reverse implication (if $h(t + \eta_i \,\cdot) \to c$ in $L^2_{loc}(\mathbb{R}, \mathbb{R}^r)$, then $\mathrm{Tan}(\gamma; t)$ contains a horizontal line) follows a similar argument, detailed more generally below.

Remark 3.13. From the point of view of the controls, a *coordinate-free* version of the tangent cone can be defined as follows. Fix a horizontal curve $\gamma : [-T, T] \to M$ parametrized by arclength, with control $h \in L^\infty([-T, T], \mathbb{R}^r)$, and a $t \in (-T, T)$. We let $\mathrm{cTan}(\gamma; t) \subseteq L^\infty(\mathbb{R}; \mathbb{R}^r)$ be the set of functions k which can be obtained as the limit, in the weak $L^2_{loc}(\mathbb{R}, \mathbb{R}^r)$-topology, of a sequence $h(t + \eta_i \cdot)$ with $\eta_i \downarrow 0$.

Arguing as in Remark 3.12 we get that, whenever $\kappa \in \mathrm{Tan}(\gamma; t)$, its control k (which uniquely determines κ) lies in $\mathrm{cTan}(\gamma; t)$. Conversely, assume that $k \in \mathrm{cTan}(\gamma; t)$, i.e., $k = \lim_{i \to \infty} h(t + \eta_i \cdot)$ for some sequence $\eta_i \downarrow 0$; up to subsequences we have

$$\delta_{1/\eta_i}\varphi(\gamma(t + \eta_i \cdot)) \to \kappa$$

for some curve $\kappa : \mathbb{R} \to M^\infty$, uniformly on compact subsets of $\mathbb{R}$ (with the same proof as Proposition 3.2), so $\kappa \in \mathrm{Tan}(\gamma;t)$ and κ has control k.

This establishes a bijective correspondence between $\mathrm{Tan}(\gamma;t)$ and $\mathrm{cTan}(\gamma;t)$. Moreover, if γ is a length minimizer, an equivalent definition of $\mathrm{cTan}(\gamma;t)$ is obtained using the strong $L^2_{loc}(\mathbb{R},\mathbb{R}^r)$-convergence (with the same proof used in Remark 3.12) and, in particular, $|k| = 1$ a.e. for all $k \in \mathrm{cTan}(\gamma;t)$.

In view of this correspondence, whenever $\mathrm{Tan}(\gamma;t)$ contains a horizontal line then this holds independently of all the choices made to construct $\mathrm{Tan}(\gamma;t)$. A similar remark holds for $\mathrm{Tan}^+(\gamma;0)$ and $\mathrm{Tan}^-(\gamma;T)$. Also, one can form the tangent cone in any system of privileged coordinates and the correspondence with $\mathrm{cTan}(\gamma;t)$ still holds, as the key inequality $|a_{ij}(\gamma(t+\eta\tau))| \le C(\eta\tau)^{w_j-1}$, established in Proposition 3.2 and needed for the precompactness of the dilated curves, is satisfied (see e.g. [10, Proposition 2.2] and recall that $d(\gamma(t+\eta\tau),\gamma(t)) \le C\eta\tau$).

4. Lifting the tangent structure to a free Carnot group

In this section we show how a tangent CC structure $(M^\infty, \mathscr{X}^\infty)$ can be lifted to a free Carnot group F, by means of a desingularization process. This is already present in the literature, see e.g. [10, Section 2.4]; however, we include here a construction also in order to show that length minimizers in M^∞ lift to length minimizers in F.

Let $(M^\infty, \mathscr{X}^\infty)$ be a tangent CC structure as in Section 2. The Lie algebra $\mathfrak{g}$ generated by $\mathscr{X}^\infty = (Y_1^\infty, \dots, Y_r^\infty)$ is nilpotent because, by Proposition 2.5, any iterated commutator of length greater than s vanishes. The identity $(\delta_\lambda)_* Y_i^\infty = \lambda Y_i^\infty$ implies that $(\delta_\lambda)_* X \to 0$ pointwise as $\lambda \to 0$, for any $X \in \mathfrak{g}$. We deduce that the j-th component of X is a polynomial function depending only on the previous variables. It follows that the flow $(x,t) \mapsto \exp(tX)(x)$ is a polynomial function in $(x,t) \in M^\infty \times \mathbb{R}$ and X is therefore complete.

Let $\mathfrak{f}$ be the free Lie algebra of rank r and step s, with generators $W_1, \dots, W_r$. The connected, simply connected Lie group F with Lie algebra $\mathfrak{f}$ can be constructed explicitly as follows: we let $F := \mathfrak{f}$ and we endow F with the group operation $A \cdot B := P(A,B)$, where

$$P(A,B) = \sum_{p=1}^{s} \frac{(-1)^{p+1}}{p} \sum_{1 \le k_i + \ell_i \le s} \frac{[A^{k_1}, B^{\ell_1}, \dots, A^{k_p}, B^{\ell_p}]}{k_1! \cdots k_p! \ell_1! \cdots \ell_p! \sum_i (k_i + \ell_i)}. \tag{4.32}$$

This is a finite truncation of the series in (2.8): the omitted terms vanish by the nilpotency of $\mathfrak{f}$. One readily checks that $P(A,0) = P(0,A) = A$ and $P(A,-A) = P(-A,A) = 0$, while the associativity identity $P(P(A,B),C) = P(A,P(B,C))$ is shown in [8, Section X.2] for free Lie algebras and can be deduced for $\mathfrak{f}$ by truncation. For any $A \in F$, $t \mapsto tA$ is a one-parameter subgroup. From this, it is straightforward to check that $\mathfrak{f}$ identifies with the Lie algebra of F, with $\exp : \mathfrak{f} \to F$ given by the identity map. In particular, $\exp : \mathfrak{f} \to F$ is a diffeomorphism and we have

$$\exp(A)\exp(B) = \exp(P(A,B)), \quad A, B \in \mathfrak{f}. \tag{4.33}$$

The group F is a *Carnot group*, which means that it is a connected, simply connected and nilpotent Lie group whose Lie algebra is stratified, i.e., it has an assigned decomposition $\mathfrak{f} = \mathfrak{f}_1 \oplus \cdots \oplus \mathfrak{f}_s$ satisfying $[\mathfrak{f}_1, \mathfrak{f}_{i-1}] = \mathfrak{f}_i$ and $[\mathfrak{f}, \mathfrak{f}_s] = \{0\}$ (in this case $\mathfrak{f}_1$ is the linear span of $W_1, \dots, W_r$). The group F just constructed is called the *free Carnot group of rank r and step s.*

Proposition 4.1. The group F is generated by $\exp(\mathfrak{f}_1)$.

Proof. See [6, Lemma 1.40]. □

By the nilpotency of $\mathfrak{g}$, there exists a unique homomorphism $\psi : \mathfrak{f} \to \mathfrak{g}$ such that $\psi(W_i) = Y_i^\infty \in \mathfrak{g}$ for $i = 1, \dots, r$. The group F acts on M^∞ on the right. The action $M^\infty \times F \to M^\infty$ is given by $(x, f) \mapsto x \cdot f := \exp(\psi(A))(x)$, where $f = \exp(A)$. In fact, by (4.33), for any $f' = \exp(B)$ we have

$$x \cdot (ff') = \exp(P(\psi(A), \psi(B)))(x) = \exp(\psi(B)) \circ \exp(\psi(A))(x) = (x \cdot f) \cdot f'. \tag{4.34}$$

The second equality is a consequence of the formula $\exp(P(tY, tX))(x) = \exp(tX) \circ \exp(tY)(x)$ for $X, Y \in \mathfrak{g}$ (with P given by (4.32)), which holds since both sides are polynomial functions in t, with the same Taylor expansion (by (2.7)). We define the map

$$\pi^\infty : F \to M^\infty, \quad \pi^\infty(f) := 0 \cdot f,$$

where the dot stands for the right action of F on M^∞.

Let $\mathscr{W} := \{W_1, \dots, W_r\}$ and extend $\mathscr{W}$ to a basis $W_1, \dots, W_N$ of $\mathfrak{f}$ adapted to the stratification. Via the exponential map $\exp : \mathfrak{f} \to F$, the one-parameter group of automorphisms of $\mathfrak{f}$ defined by $W_k \mapsto \lambda^i W_k$ if and only if $W_k \in \mathfrak{f}_i$ induces a one-parameter group of automorphisms $(\widehat{\delta}_\lambda)_{\lambda>0}$ of F, called *dilations.*

If $A \in \mathfrak{f}_1$, for any $\lambda > 0$ and $x \in M^\infty$ we have the identity

$$\exp(\lambda\psi(A))(\delta_\lambda(x)) = \delta_\lambda\big(\exp(\psi(A))(x)\big), \tag{4.35}$$

which follows from $(\delta_\lambda)_*\psi(A) = \lambda\psi(A)$.

Definition 4.2. We call the CC structure $(F, \mathscr{W})$ the *lifting* of $(M^\infty, \mathscr{X}^\infty)$ with projection $\pi^\infty : F \to M^\infty$.

Proposition 4.3. The lifting $(F, \mathscr{W})$ of $(M^\infty, \mathscr{X}^\infty)$ has the following properties:

(i) for any $f \in F$ and $i = 1, \dots, r$ we have $\pi^\infty_*(W_i(f)) = Y_i^\infty(\pi^\infty(f))$;

(ii) the dilations of F and M^∞ commute with the projection: namely, for any $\lambda > 0$ we have

$$\pi^\infty \circ \widehat{\delta}_\lambda = \delta_\lambda \circ \pi^\infty.$$

Proof. (i) Using the action property (4.34), we find

$$\begin{aligned}\pi^\infty_*(W_i(f)) &= \frac{d}{dt}\pi^\infty\big(f\exp(tW_i)\big)\Big|_{t=0} = \frac{d}{dt}0\cdot\big(f\exp(tW_i)\big)\Big|_{t=0}\\ &= \frac{d}{dt}\pi^\infty(f)\cdot\exp(tW_i)\Big|_{t=0} = \psi(W_i)(\pi^\infty(f)) = Y_i^\infty(\pi^\infty(f)).\end{aligned}$$

(ii) Let $\lambda>0$ and $x\in M^\infty$. By (4.35), for any $W\in\mathfrak{f}_1$ we have

$$\begin{aligned}\delta_\lambda(x)\cdot\exp(\lambda W) &= \exp(\lambda\psi(W))(\delta_\lambda(x)) = \delta_\lambda\big(\exp(\psi(W))(x)\big)\\ &= \delta_\lambda(x\cdot\exp(W)).\end{aligned}\tag{4.36}$$

We deduce that the claim holds for any $f=\exp(W)$ with $W\in\mathfrak{f}_1$, because

$$\pi^\infty(\widehat{\delta}_\lambda(f)) = \pi^\infty(\exp(\lambda W)) = \delta_\lambda(0)\cdot\exp(\lambda W) = \delta_\lambda(0\cdot\exp(W)) = \delta_\lambda(\pi^\infty(f)).$$

By Proposition 4.1, any $f\in F$ is of the form $f=f_1f_2\dots f_k$ with each $f_i\in\exp(\mathfrak{f}_1)$. Assume by induction that the claim holds for $\widehat{f}=f_1f_2\dots f_{k-1}$. By (4.36), letting $f_k=\exp(W)$ we have

$$\begin{aligned}\pi^\infty(\widehat{\delta}_\lambda(f)) &= \pi^\infty(\widehat{\delta}_\lambda(\widehat{f})\exp(\lambda W)) = \pi^\infty(\widehat{\delta}_\lambda(\widehat{f}))\cdot\exp(\lambda W)\\ &= \delta_\lambda(\pi^\infty(\widehat{f}))\cdot\exp(\lambda W) = \delta_\lambda\big(\pi^\infty(\widehat{f})\cdot\exp(W)\big) = \delta_\lambda\big(\pi^\infty(f)\big).\end{aligned}$$ □

Let $\kappa: I\to M^\infty$ be a horizontal curve in $(M^\infty,\mathscr{X}^\infty)$, with control $h\in L^\infty(I,\mathbb{R}^r)$. A horizontal curve $\overline{\kappa}: I\to F$ such that

$$\kappa = \pi^\infty\circ\overline{\kappa} \qquad \text{and} \qquad \dot{\overline{\kappa}}(t) = \sum_{i=1}^r h_i(t)W_i(\overline{\kappa}(t)) \quad \text{for a.e. } t\in I$$

is called a *lift* of κ to $(F,\mathscr{W})$.

Proposition 4.4. Let $(F,\mathscr{W})$ be the lifting of $(M^\infty,\mathscr{X}^\infty)$ with projection $\pi^\infty: F\to M^\infty$. Then the following facts hold:

(i) If κ is length minimizing in $(M^\infty,\mathscr{X}^\infty)$, then any horizontal lift $\overline{\kappa}$ of κ is length minimizing in $(F,\mathscr{W})$.

(ii) If $\overline{\kappa}$ is a horizontal (half-)line in F, then $\pi^\infty\circ\overline{\kappa}$ is a horizontal (half-)line in $(M^\infty,\mathscr{X}^\infty)$.

Proof. Claim (i) follows from $L(\overline{\kappa})=L(\kappa)$ and from the inequality $L(\overline{\kappa}')=L(\kappa')\geq L(\kappa)$, whenever $\overline{\kappa}'$ is horizontal with the same endpoints as $\overline{\kappa}$ and $\kappa'=\pi^\infty\circ\overline{\kappa}'$. We now turn to claim (ii). Let $\overline{\kappa}(t)=\exp(tW)$ for some $W\in\mathfrak{f}_1$. The projection $\pi^\infty\circ\overline{\kappa}$ is horizontal by part (i) of Proposition 4.3. The thesis follows from characterization (i) for horizontal lines, contained in Proposition 3.8. □

References

[1] A. A. Agrachev, D. Barilari & U. Boscain, *Introduction to Riemannian and sub-Riemannian geometry*, available at https://webusers.imj-prg.fr/~davide.barilari/Notes.php.

[2] A. Bellaïche, *The tangent space in sub-Riemannian geometry.* In *Sub-Riemannian Geometry*, Progress in Mathematics 144 (1996), 1–78.

[3] C. Carathéodory, *Untersuchungen über die Grundlagen der Thermodynamik.* Math. Ann. 67 (1909), 355–386.

[4] W. L. Chow, *Über Systeme von linearen partiellen Differentialgleichungen erster Ordnung.* Math. Ann. 117 (1939), 98–105.

[5] S. Don & D. Vittone, *Fine properties of functions with bounded variation in Carnot–Carathéodory spaces.* In preparation.

[6] G. B. Folland & E. M. Stein, *Hardy spaces on homogeneous groups.* Mathematical Notes, 28. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1982.

[7] H. Hermes, *Nilpotent and High-Order Approximations of Vector Field Systems.* SIAM Rev. 33 (1991), no. 2, 238–264.

[8] G. P. Hochschild, *The structure of Lie groups.* Holden–Day series in mathematics. Holden–Day Inc., San Francisco, London, Amsterdam, 1965.

[9] E. Hakavuori & E. Le Donne, *Non-minimality of corners in subriemannian geometry.* Invent. Math. 206 (2016), no. 3, 693–704.

[10] F. Jean, *Control of Nonholonomic Systems: from Sub-Riemannian Geometry to Motion Planning.* Springer Briefs in Mathematics. Springer, Cham, 2014.

[11] G. P. Leonardi & R. Monti, *End-point equations and regularity of sub-Riemannian geodesics.* Geom. Funct. Anal. 18 (2008), no. 2, 552–582.

[12] G. A. Margulis & G. D. Mostow, *Some remarks on the definition of tangent cones in a Carnot–Carathéodory space.* J. Anal. Math. 80 (2000), no. 1, 299–317.

[13] J. Mitchell, *On Carnot–Carathéodory metrics.* J. Differ. Geom. 21 (1985), 35–45.

[14] R. Monti, A. Pigati & D. Vittone, *Existence of tangent lines to Carnot–Carathéodory geodesics.* Calc. Var. Partial Differential Equations (2018) 57:75, online.

[15] A. Nagel, E. M. Stein & S. Wainger, *Balls and metrics defined by vector fields I: Basic properties.* Acta Math. 155 (1985), no. 1, 103–147.

[16] P. K. Rashevsky, *Any two points of a totally nonholonomic space may be connected by an admissible line.* Uch. Zap. Ped. Inst. im. Liebknechta, Ser. Phys. Math. 2 (1938), 83–94.

[17] L. Rifford, *Singulières minimisantes en géométrie sous-riemannienne.* Séminaire Bourbaki, Mars 2016, 68ème année, 2015–2016, n. 1113.

E-mail address: `monti@math.unipd.it`

E-mail address: `alessandro.pigati@math.ethz.ch`

E-mail address: `vittone@math.unipd.it`

(Monti and Vittone) Università di Padova, Dipartimento di Matematica "T. Levi-Civita", via Trieste 63, 35121 Padova, Italy

(Pigati) ETH Zürich, Department of Mathematics, Rämistrasse 101, 8092 Zürich, Switzerland