A PROOF OF THE MULTIPLICITY ONE CONJECTURE FOR MIN-MAX MINIMAL SURFACES IN ARBITRARY CODIMENSION

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Abstract. Given any admissible k-dimensional family of immersions of a given closed oriented surface into an arbitrary closed Riemannian manifold, we prove that the corresponding min-max width for the area is achieved by a smooth (possibly branched) immersed minimal surface with multiplicity one and Morse index bounded by k.

1. Introduction

Recently, a new theory for the construction of branched immersed minimal surfaces of arbitrary topology, in an assigned closed Riemannian manifold $\mathcal{M}^m$, was proposed in [12]. This method is based on a penalization of the area functional by means of the second fundamental form A of the immersion.

Namely, for a fixed parameter $\sigma > 0$, one first finds an immersion $\Phi : \Sigma \to M$ which is critical for the perturbed area functional

$$A^\sigma(\Phi) := \int_\Sigma d\mathrm{vol}_{g_\Phi} + \sigma^2 \int_\Sigma (1 + |A|^2_{g_\Phi})^2 \, d\mathrm{vol}_{g_\Phi}, \tag{11}$$

where Σ is a fixed closed oriented surface and g_Φ is the metric induced by Φ, with volume form vol_{g_Φ}. This functional A^σ enjoys a sort of Palais–Smale condition up to diffeomorphisms.

We should mention that the idea of considering perturbed functionals goes back to the paper [16] by Sacks–Uhlenbeck, where a perturbation of the Dirichlet energy is used to build minimal immersed spheres. However, in order to find minimal immersed surfaces with higher genus, one should give up working with the Dirichlet energy and use a more tensorial functional like (11): among closed orientable surfaces, only the sphere has a unique conformal structure (up to diffeomorphisms) and, as a consequence, a harmonic map (i.e. a critical point for the Dirichlet energy) $\Phi : \Sigma \to \mathcal{M}^m$ could fail to be conformal and minimal if Σ has positive genus. In principle, one can overcome this issue by introducing the conformal structure as an additional parameter in the variational problem: this program was carried over by Zhou in [19].

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Considering any sequence $\sigma_j \downarrow 0$, one gets a sequence $\Phi_j : \Sigma_j \to M$ of conformal immersions (with area bounded above and below), where Σ_j denotes Σ endowed with the induced conformal structure. Assuming for simplicity that we are dealing with a constant conformal structure (in general one gets a limiting Riemann surface in the Deligne–Mumford compactification), the sequence Φ_j is then bounded in $W^{1,2}$ and we can consider its weak limit Φ_∞, up to subsequences. A priori it is not clear whether the strong $W^{1,2}$-convergence holds, even away from a finite bubbling set. However, in [12] the second author shows that, if the sequence σ_j is carefully chosen so as to satisfy a certain *entropy condition*, then the surfaces $\Phi_j(\Sigma_j)$ converge to a *parametrized stationary varifold* (a notion introduced in [12, 11] and recalled in Section 3 below) which we call $(\Sigma_\infty, \Theta_\infty, N_\infty)$ in the present paper. The limiting multiplicity N_∞ a priori could be bigger than one.

A consequence of the main regularity result contained in [11] is that the multiplicity N_∞ is locally constant.

This result, which is optimal for the class of parametrized stationary varifolds, leaves nonetheless open the question whether one can have $N_\infty > 1$ on some connected component of Σ_∞.

This question should be compared with the *multiplicity one conjecture* by Marques and Neves. In [9], the following upper bound for the Morse index of a minimal hypersurface with locally constant multiplicity is established: if

$$\Sigma = \sum_{j=1}^{\ell} n_j \Sigma_j$$

is a minimal hypersurface with locally constant multiplicity, given by a min-max with k parameters in the context of Almgren–Pitts theory, then

$$\operatorname{index}(\operatorname{supp}(\Sigma)) \le k, \qquad \operatorname{supp}(\Sigma) := \bigsqcup_{j=1}^{\ell} \Sigma_j.$$

In other words, this is a bound for the Morse index of the hypersurface obtained by replacing all the multiplicities n_j with 1. In order for this estimate to give more information about Σ, or at least its unstable part, the authors make the following conjecture.

Conjecture 1.1 (Multiplicity one conjecture). *For generic metrics on M^{n+1}, with $3 \le n+1 \le 7$, two-sided unstable components of closed minimal hypersurfaces obtained by min-max methods must have multiplicity one.*

It is natural to demand for extra information for one-sided stable components with unstable double cover, as well, even if this situation is expected not to show up generically.

Marques and Neves were able to prove this conjecture for one-parameter sweepouts, but the general case remains open. For metrics with positive Ricci curvature, the one-parameter case was already discussed by Marques and Neves in [8] and later by Zhou in [18].

Further results, such as the two-sidedness of Σ when the metric has positive Ricci curvature, were obtained by Ketover, Marques and Neves in [6], using the catenoid estimate.

We also mention that Ketover, Liokumovich and Song in [5, 17] started to settle the generic, one-parameter case for the simpler and more effective Simon–Smith variant of Almgren–Pitts theory, specially designed for 3-manifolds.

Very recently, in [1], the conjecture was established for bumpy metrics in 3-manifolds, i.e. when $n = 2$, in the setting of Allen–Cahn level set approach.

The importance of this conjecture in relation to the Morse index of Σ is twofold. First of all, there is no satisfactory definition for the Morse index of an embedded minimal hypersurface with multiplicity bigger than one: such Σ could be thought as the limiting object of many qualitatively different sequences, e.g. the elements of the sequence could realize different covering spaces of the limit, or more pathologically they could have many catenoidal necks (hence Σ would be the limit of a sequence of highly unstable hypersurfaces).

Also, if one is able to establish a lower bound on the Morse index such as

$$k \leq \text{index}(\text{supp}\,(\Sigma)) + \text{nullity}(\text{supp}\,(\Sigma)),$$

then the multiplicity one conjecture gives infinitely many *geometrically distinct* minimal hypersurfaces, provided there exists at least one for every value of k. This was precisely the strategy used in [1] to prove Yau's conjecture for generic metrics: in [1] the authors obtained the multiplicity one result and the equality $\text{index}(\Sigma) = k$ (the nullity vanishing automatically for bumpy metrics).

In this work we establish the natural counterpart of this conjecture in our setting, namely for minimal surfaces produced by the viscous relaxation method.

Theorem 1.2. *We have $N_\infty \equiv 1$.*

We stress that this result holds in *arbitrary codimension* and *without any genericity assumption.*

We remark that, in view of earlier work in [13], this statement would imply by itself the main result of [11], for parametrized stationary varifolds arising as a limit of stationary points for the relaxed functionals. However, the proof of Theorem 1.2 relies substantially on the regularity result obtained in [11], needed in several compactness arguments.

The main idea is to define a sort of *macroscopic multiplicity*, on balls $B^q_\ell(p)$, before passing to the limit (i.e. looking at the immersed surfaces Φ_j rather than their limit). Then we will use a continuity argument to show that this number stays constant as we pass from scale 1 to scale $\sqrt{\sigma_j}$. At the latter scale we have a very clear understanding of the behaviour of Φ_j and in particular we are able to say that here the macroscopic multiplicity equals 1. Thus the same holds at the original scale and this is sufficient to get $N_\infty \equiv 1$.

Corollary 1.3. *If there is no bubbling or degeneration of the underlying conformal structure, we have strong $W^{1,2}$-convergence $\Phi_k \to \Phi_\infty$. In general we have a bubble tree convergence.*

Theorem 1.2 and Corollary 1.3 pave the way to obtain meaningful Morse index bounds. Indeed, although Theorem 1.2 does not rule out the possibility of having a surface covered multiple times by Φ_∞, a crucial advantage of having a parametrization at our disposal is that we have a reasonable definition of Morse index and nullity: they are defined for the area functional, with respect to variations in $C_c^\infty(\Sigma_\infty \setminus \{z_1, \dots, z_s\})$, the points $z_1, \dots, z_s$ being the branch points of the immersion Φ_∞. [1]

The natural expected inequalities would be

$$\text{index}(\Phi_\infty) \le k \le \text{index}(\Phi_\infty) + \text{nullity}(\Phi_\infty).$$

An abstract framework to show upper bounds for the Morse index, dealing with general penalized functionals on Banach manifolds, is developed in [10]. Combining Corollary 1.3 with the general result obtained in [10] and with [14], we reach the following conclusion (we refer the reader to [10] for the notion of *admissible family*).

Corollary 1.4. *Given an admissible family $\mathcal{A} \subseteq \mathcal{P}(Imm(\Sigma, \mathcal{M}^m))$ of dimension k and calling*

$$W_{\mathcal{A}} := \inf_{A \in \mathcal{A}} \sup_{\Phi \in A} \text{area}(\Phi)$$

the width *of $\mathcal{A}$, there exists a (possibly branched) minimal immersion Φ of a closed surface S into $\mathcal{M}^m$ such that*

(i) $\text{genus}(S) \le \text{genus}(\Sigma)$,
(ii) $W_{\mathcal{A}} = \text{area}(\Phi)$,
(iii) $\text{index}(\Phi) \le k$.

However, proving the second inequality, namely $k \le \text{index}(\Phi) + \text{nullity}(\Phi)$, seems to require a finer understanding of the convergence $\Phi_k \to \Phi_\infty$. We hope to be able to deal with this question elsewhere.

Also, it would be interesting to adapt the well-known approach based on Gromov–Guth *p-width* $\omega_p(\mathcal{M})$ (or higher codimension generalizations), used to produce infinitely many minimal hypersurfaces in many settings, to the present situation. To this aim, a natural topological question concerns how much *genus* is needed to realize a nontrivial p-sweepout (in the sense of Gromov–Guth), and how to realize the sweepout within the space of immersions.

[1] Although we are dealing with a weakly conformal map Φ_∞, for which area and energy are the same, it is important to remark that the Morse indexes for area and energy, denoted index_A and index_E respectively, should not be expected to agree. The relationship between the two is a subtle problem: in this direction, we mention the inequality $\text{index}_E(\Psi) \le \text{index}_A(\Psi) \le \text{index}_E(\Psi) + r$ established in [2], for a branched minimal immersion Ψ, where $r = r(g, b)$ depends on the genus and the number of branch points of Ψ.

2. Notation

- We will assume, without loss of generality, that $\mathcal{M}^m$ is isometrically embedded in some Euclidean space $\mathbb{R}^q$. Given $p \in \mathcal{M}^m$ and $\ell > 0$, we set $\mathcal{M}^m_{p,\ell} := \ell^{-1}(\mathcal{M}^m - p)$.
- In what follows, Π will always denote a 2-plane through the origin, which we identify with the corresponding orthogonal projection $\Pi : \mathbb{R}^q \to \Pi$. We call $\Pi^\perp$ the orthogonal $(q-2)$-subspace, identified with the corresponding orthogonal projection. Given 2-planes Π, Π', their distance $\operatorname{dist}(\Pi, \Pi')$ is the one induced by the Plücker's embedding of the Grassmannian $\mathrm{Gr}_2(\mathbb{R}^q)$ into the projectivization of $\Lambda_2\mathbb{R}^q$.

  The adjoint maps, which are just the inclusions $\Pi \hookrightarrow \mathbb{R}^q$ and $\Pi^\perp \hookrightarrow \mathbb{R}^q$, are denoted Π^* and $(\Pi^\perp)^*$, so that

$$\mathrm{id}_{\mathbb{R}^q} = \Pi^*\Pi + (\Pi^\perp)^*\Pi^\perp. \tag{21}$$

  Also, Π_0 is the canonical 2-plane, so that $\Pi_0 : \mathbb{R}^q \to \mathbb{R}^2$ is the projection onto the first two coordinates, while $\Pi_0^\perp : \mathbb{R}^q \to \mathbb{R}^{q-2}$ is the projection onto the remaining $q-2$.
- We call $B^2_r(x)$ the ball of center x and radius r in the plane $\mathbb{C} = \mathbb{R}^2$, while $B^q_s(p)$ will denote the ball of center p and radius s in $\mathbb{R}^q$. Given $p \in \Pi$, we call $B^\Pi_s(p)$ the two-dimensional ball with center p and radius s in Π, i.e. $B^\Pi_s(p) := B^q_s(p) \cap \Pi$. When the center is not specified, it is always meant to be the origin.
- Given a function $\Psi \in W^{1,2}(B^2_r(x))$ and $0 < s \le r$, the notation $\Psi\big|_{\partial B^2_s(x)}$ always refers to the trace of Ψ on the circle $\partial B^2_s(x)$.
- Given $K \ge 1$, we define the following set of Beltrami coefficients:

$$\mathcal{E}_K := \left\{ \mu \in L^\infty(\mathbb{C}, \mathbb{C}),\ \|\mu\|_{L^\infty} \le \frac{K-1}{K+1} \right\}. \tag{22}$$

  We let $\mathcal{D}_K$ denote the set of K-quasiconformal homeomorphisms $\varphi : \mathbb{C} \to \mathbb{C}$ such that

$$\varphi(0) = 0, \qquad \min_{x \in \partial B^2_1} |\varphi(x)| = 1. \tag{23}$$

  We have $\varphi \in W^{1,2}_{loc}(\mathbb{C})$ and $\partial_z\varphi = \mu\partial_{\bar z}\varphi$ for some $\mu \in \mathcal{E}_K$, in the weak sense; we refer the reader to [4, Chapter 4] for the basic theory of K-quasiconformal homeomorphisms in the plane. Moreover, φ is a linear map in $\mathcal{D}_K$ if and only if $\varphi(e_1) = e_1'$ and $\varphi(e_2') = \lambda e_2'$, for suitable orthonormal bases (e_1, e_2), (e_1', e_2') inducing the canonical orientation and a suitable $1 \le \lambda \le K$.
- We define

$$D(K) := \sup\left\{ |\varphi(x)| \,;\, x \in \overline{B}^2_1, \varphi \in \mathcal{D}_K \right\}, \quad s(K) := \inf\left\{ \left|\varphi^{-1}(y)\right| ; |y| \ge \frac{1}{2}, \varphi \in \mathcal{D}_K \right\},$$

  so that $\varphi(\overline{B}^2_1) \subseteq \overline{B}^2_{D(K)}$ and $\varphi(\overline{B}^2_{s(K)}) \subseteq \overline{B}^2_{1/2}$ for all $\varphi \in \mathcal{D}_K$. The fact that $D(K) < \infty$ and $s(K) > 0$ is guaranteed by Lemma A.4. We also set

$$\eta(K) := \frac{1}{4} \inf\left\{ |\varphi(x)| \,;\, x \in \partial B^2_{s(K)^2}, \varphi \in \mathcal{D}_K \right\} > 0.$$

- We let $\mathcal{D}_K^\Pi$ denote the set of maps having the form $\Pi^* \circ R \circ \varphi$, where $\varphi \in \mathcal{D}_K$ and $R : \mathbb{R}^2 \to \Pi$ is a linear isometry. Given $0 < \delta < 1$, we call $\mathcal{R}_{K,\delta}^\Pi$ the set of maps in $W^{1,2}(B_1^2, \mathbb{R}^q)$ which are close to some $\psi \in \mathcal{D}_K^\Pi$ on the circles of radii $1, s(K), s(K)^2$, namely we set

(24)
$$\mathcal{R}_{K,\delta}^\Pi := \left\{ \Psi \in W^{1,2}(B_1^2, \mathbb{R}^q) : \min_{\psi \in \mathcal{D}_K^\Pi} \max_{r \in \{1, s(K), s(K)^2\}} \left\| \Psi\big|_{\partial B_r^2}(r\cdot) - \psi(r\cdot) \right\|_{L^\infty(\partial B_1^2)} \le \delta \right\}.$$

- Given $\Psi \in C^1(\Omega, \mathbb{R}^q)$, a ball $B_r^2(z) \subset\subset \Omega$ and a 2-plane Π, we define the *projected multiplicity*

$$N_{\Psi,z,r}^\Pi : \Pi \to \mathbb{N} \cup \{\infty\}, \qquad N_{\Psi,z,r}^\Pi(p) := \#(\Pi \circ \Psi)^{-1}(p) \cap B_r^2(z) \tag{25}$$

and, given $p \in \mathbb{R}^q$, we also define the *macroscopic multiplicity*

$$n_{\Psi,z,r}^{\Pi,p,t} := \left\lfloor \fint_{B_t^\Pi(\Pi(p))} N_{\Psi,z,r}^\Pi \right\rfloor \in \mathbb{N}. \tag{26}$$

The mean appearing in (26) is finite by the area formula and $\lfloor \cdot \rfloor$ denotes the integer part.

3. Background on parametrized stationary varifolds

Assume we have a smooth conformal map $\Phi : B_1^2 \to \mathcal{M}^m$, critical for the functional

$$\Phi \mapsto \int_{B_1^2} d\mathrm{vol}_{g_\Phi} + \sigma^2 \int_{B_1^2} (1 + |A_{g_\Phi}|_{g_\Phi}^2)^2 \, d\mathrm{vol}_{g_\Phi}, \tag{31}$$

and assume that the following *entropy condition*

$$\sigma^2 \log(\sigma^{-1}) \int_{B_1^2} (1 + |A|^2)^2 \, d\mathrm{vol}_{g_\Phi} \le \varepsilon \int_{B_1^2} d\mathrm{vol}_{g_\Phi} \tag{32}$$

holds for some $\varepsilon > 0$. Notice that the second integral equals $\frac{1}{2} \int_{B_1^2} |\nabla \Phi|^2$.

Given any $0 < \ell < 1$ and $p \in \mathcal{M}^m$, the rescaled map

$$\Psi : B_1^2 \to \mathcal{M}_{p,\ell}^m, \qquad \Psi := \ell^{-1}(\Phi - p)$$

is critical for the functional

$$\int_{B_1^2} d\mathrm{vol}_{g_\Psi} + \tau^2 \int_{B_1^2} (\ell^2 + |A|^2)^2 \, d\mathrm{vol}_{g_\Psi}, \qquad \tau := \sigma \ell^{-2} \tag{33}$$

and, being $\tau^2 \log(\tau^{-1}) \le \ell^{-4} \sigma^2 \log(\sigma^{-1})$, it satisfies

$$\tau^2 \log(\tau^{-1}) \int_{B_1^2} (\ell^2 + |A|^2)^2 \, d\mathrm{vol}_{g_\Psi} \le \varepsilon \int_{B_1^2} d\mathrm{vol}_{g_\Psi}, \tag{34}$$

where now A denotes the second fundamental form of Ψ in $\mathcal{M}_{p,\ell}^m$ and its norm is meant with respect to the induced metric g_Ψ.

In the sequel, we will establish many intermediate results on maps Ψ arising in this way, by means of compactness arguments. The starting point in these arguments is that, if we have sequences Ψ_k, p_k, $\ell_k \to 0$, $\tau_k \to 0$ and $\varepsilon_k \to 0$, then by (33) and (34) Ψ_k should have a limit point Ψ_∞ (in some weak sense) which is critical for the area functional in the tangent space $T_{p_\infty}\mathcal{M}^m$ (where p_∞ is a limit point of the sequence p_k), i.e. Ψ_∞ should be a minimal parametrization.

Indeed, invoking previous work from [12] and [11], we get that up to subsequences we have convergence to a (local) parametrized stationary varifold, whose definition is recalled below, restricting for simplicity to the case (sufficient for the purposes of this paper) where the ambient manifold equals $\mathbb{R}^q$. A rigorous explanation of the kind of convergence taking place is given in Remark 5.3 below.

Definition 3.1. A triple (Σ, Φ, N), with Σ a closed connected Riemann surface, $\Phi \in W^{1,2}(\Sigma, \mathbb{R}^q)$ nonconstant, weakly conformal and $N \in L^\infty(\Sigma, \mathbb{N} \setminus \{0\})$, is a *parametrized stationary varifold* if for almost every $\omega \subseteq \Sigma$ the 2-rectifiable varifold

$$\mathbf{v}_\omega := (\Phi(\mathcal{G} \cap \omega), \theta_\omega), \qquad \theta_\omega(p) := \sum_{x \in \mathcal{G} \cap \omega \cap \Phi^{-1}(p)} N(x)$$

is stationary in the open set $\mathbb{R}^q \setminus \Phi(\partial\omega)$, where $\mathcal{G}$ denotes the set of Lebesgue points for both Φ and $d\Phi$.

We refer the reader to [11, Definition 2.1] for the notion of *almost every domain*, as well as to [11, Definition 2.2] for another definition, whose equivalence with Definition 3.1 is detailed in [11, Remark 2.3]. The latter formulation will not be used here.

Also, there is a corresponding local notion where we have an open set $\Omega \subseteq \mathbb{C}$ in place of Σ and where we require the stationarity condition for a.e. $\omega \subset\subset \Omega$: see [11, Definition 2.9]. This is the notion mostly used in this paper.

As already mentioned in the introduction, the main result of [11] is that Φ is harmonic (i.e. coincides a.e. with a harmonic map) and N is (a.e.) constant; in the local version, this holds on the connected components of Ω where Φ is not (a.e.) constant.

4. Two lemmas on harmonic maps

Lemma 4.1. *Let $\gamma_k \in C^0(\partial B_1^2, \mathbb{R}^2)$ be a sequence of Jordan curves converging (in C^0) to a Jordan curve γ_∞ and let $f_k \in C^0(\partial B_1^2)$ be a sequence converging uniformly to a function f_∞. Let D_k be the domain bounded by γ_k, let $u_k \in C^0(\overline{D}_k)$ be the harmonic extension of $f_k \circ \gamma_k^{-1}$, and similarly define D_∞ and u_∞. Then $u_k \to u_\infty$ in $C^0_{loc}(D_\infty)$. Moreover, if $y_k \to y_\infty$ with $y_k \in \overline{D}_k$ and $y_\infty \in \overline{D}_\infty$, then $u_k(y_k) \to u_\infty(y_\infty)$.*

Notice that such harmonic extensions exist and are unique since there exist homeomorphisms $\overline{B}_1^2 \to \overline{D}_k$ restricting to biholomorphisms $B_1^2 \to D_k$ (and similarly for D_∞).

Proof. Since the functions f_k are equibounded, from the maximum principle and interior estimates it follows that the functions u_k are equibounded in $C^2(\overline{\omega})$, for any $\omega \subset\subset D_\infty$, and hence by Ascoli–Arzelà theorem the convergence $u_k \to u_\infty$ in $C^0_{loc}(D_\infty)$ follows from the second claim.

It suffices to show that the second claim holds for a subsequence: once this is done, it can be obtained for the full sequence by a standard contradiction argument (given a sequence $y_k \to y_\infty$, if $u_k(y_k)$ did not converge to $u_\infty(y_\infty)$, we could find a subsequence such that it converges to a different value; then we would reach a contradiction along a further subsequence where the second claim holds).

Up to removing a finite set of indices, we can suppose that there is a point p such that $p \in D_k$ for all $k \in \mathbb{N} \cup \{\infty\}$. By Carathéodory's theorem, we can find homeomorphisms $\upsilon_k : B_1^2 \to \overline{D}_k$ restricting to biholomorphisms from B_1^2 to D_k, so that $\upsilon_k\big|_{\partial B_1^2} = \gamma_k \circ \beta_k$, for suitable homeomorphisms $\beta_k : \partial B_1^2 \to \partial B_1^2$ (for all $k \in \mathbb{N}$).

Since the maps υ_k and υ_k^{-1} are equibounded and harmonic, we can assume that

$$\upsilon_k \to \upsilon_\infty, \qquad \zeta_k := \upsilon_k^{-1} \to \zeta_\infty \tag{41}$$

in $C^\infty_{loc}(B_1^2)$ and $C^\infty_{loc}(D_\infty)$, respectively. Notice that υ_∞ is a holomorphic map taking values into $\overline{D}_\infty$, while $\widetilde{\upsilon}_\infty$ is holomorphic and takes values into B_1^2 (by the maximum principle, since $\widetilde{\upsilon}_\infty(p) = 0$ and $|\upsilon_\infty| \le 1$). So for any $w \in D_\infty$ the set $\left\{\upsilon_k^{-1}(w) \mid k \in \mathbb{N}\right\} \cup \{\widetilde{\upsilon}_\infty(w)\} \subset B_1^2$ is compact and we infer

$$\upsilon_\infty \circ \zeta_\infty(w) = \lim_{k \to \infty} \upsilon_k \circ \zeta_k(w) = w. \tag{42}$$

Hence υ_∞ is surjective and thus an open map. So $\upsilon_\infty(B_1^2) = D_\infty$ and, by [15, Theorem 10.43] (applied with $f := \upsilon_\infty - w$, $g := \upsilon_k - w$, for a fixed $w \in D_\infty$ and an arbitrary circle $\partial B_r^2 \subseteq B_1^2$ avoiding $f^{-1}(w)$, with k large enough), it is also injective. By Carathéodory's theorem, it extends continuously to a homeomorphism (still denoted υ_∞) from B_1^2 to D_∞ and we have $\upsilon_\infty\big|_{\partial B_1^2} = \gamma_\infty \circ \beta_\infty$ for a suitable homeomorphism $\beta_\infty : \partial B_1^2 \to \partial B_1^2$.

Up to subsequences, applying Helly's selection principle (to the lifts $\overline{\beta}_k : \mathbb{R} \to \mathbb{R}$), we can assume that $\beta_k \to \widetilde{\beta}_\infty$ everywhere, for some order-preserving $\widetilde{\beta}_\infty$. On the other hand, since $\sup_k \int_{B_1^2} |\upsilon_k'|^2 = \sup_k \mathcal{L}^2(D_k)$ is finite, we have weak convergence $\upsilon_k \rightharpoonup \upsilon_\infty$ in $W^{1,2}(B_1^2)$ and thus weak convergence $\gamma_k \circ \beta_k \rightharpoonup \gamma_\infty \circ \beta_\infty$ in $L^2(\partial B_1^2)$. The everywhere convergence $\gamma_k \circ \beta_k \to \gamma_\infty \circ \widetilde{\beta}_\infty$ implies $\gamma_\infty \circ \beta_\infty = \gamma_\infty \circ \widetilde{\beta}_\infty$ a.e. and thus $\beta_\infty = \widetilde{\beta}_\infty$ a.e. Since β_∞ is continuous and both maps are order-preserving, we conclude that $\beta_\infty = \widetilde{\beta}_\infty$ everywhere. Using again the continuity of β_∞, as well as the everywhere convergence of the order-preserving maps $\beta_k \to \beta_\infty$, we also get that $\beta_k \to \beta_\infty$ uniformly.

Being υ_k the harmonic extension of $\gamma_k \circ \beta_k$ (for $k \in \mathbb{N} \cup \{\infty\}$), we conclude that $\upsilon_k \to \upsilon_\infty$ in $C^0(\overline{B}_1^2)$. Let $U_k \in C^0(\overline{B}_1^2)$ be the harmonic extension of $f_k \circ \beta_k$ and notice that $U_k \to U_\infty$ in $C^0(\overline{B}_1^2)$. By conformal invariance, $u_k := U_k \circ \upsilon_k^{-1}$ is the harmonic extension of $f_k \circ \gamma_k^{-1}$ on D_k (for $k \in \mathbb{N} \cup \{\infty\}$).

Finally, we claim that in the situation of the second claim we have $\upsilon_k^{-1}(y_k) \to \upsilon_\infty^{-1}(y_\infty)$. This easily follows from the injectivity of υ_∞: if we had $\left|\upsilon_k^{-1}(y_k) - \upsilon_\infty^{-1}(y_\infty)\right| \ge \varepsilon$ along some subsequence (for some $\varepsilon > 0$), we would have a limit point $x_\infty \in \overline{B}_1^2$ with $\left|x_\infty - \upsilon_\infty^{-1}(y_\infty)\right| \ge$

ε and $v_\infty(x_\infty) = \lim_{k\to\infty} y_k = y_\infty$, which is a contradiction. Hence,

$$u_k(y_k) = U_k(v_k^{-1}(y_k)) \to U_\infty(v_\infty^{-1}(y_\infty)) = u_\infty(y_\infty), \tag{43}$$

as desired. □

Lemma 4.2. *Given $K \geq 1$ and $s, \varepsilon > 0$, there exists a constant $0 < \delta_0 < \varepsilon$, depending only on q, K, s, ε, with the following property: whenever*

- *$\Psi \in W^{1,2} \cap C^0(\overline{B}_1^2, \mathbb{R}^q)$ has $\left\| \Psi\big|_{\partial B_1^2} - \psi(s\cdot)\big|_{\partial B_1^2} \right\|_{L^\infty(\partial B_1^2)} \leq \delta_0$ for some $\psi \in \mathcal{D}_K^\Pi$,*
- *$\Psi \circ \varphi^{-1}$ is harmonic and weakly conformal on $\varphi(B_1^2)$, where $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ is the normal solution to a Beltrami differential equation with a coefficient $\mu \in \mathcal{E}_K$ (in the sense of [4, Theorem 4.24]),*

then $\Pi \circ \Psi \circ \varphi^{-1}$ is a diffeomorphism from $\varphi(\overline{B}_{1/2}^2)$ onto its image, with

$$\mathrm{dist}(\Pi, \Pi(x)) < \varepsilon, \qquad \Pi(x) := \textit{2-plane spanned by } \nabla(\Psi \circ \varphi^{-1}), \tag{44}$$

and so $\Pi \circ \Psi$ is injective on $\overline{B}_{1/2}^2$.

Proof. Assume by contradiction that, for a sequence $\delta_k \downarrow 0$, there exist maps $\Psi_k : B_1^2 \to \mathbb{R}^q$, planes Π_k, homeomorphisms $\varphi_k : \mathbb{R}^2 \to \mathbb{R}^2$ and coefficients μ_k such that the claim fails with $\delta_0 = \delta_k$. By Lemma A.4, up to subsequences we have $\Pi_k \to \Pi_\infty$ and $\Psi_k\big|_{\partial B_1^2} \to \gamma$, where $\gamma : \partial B_1^2 \to \mathbb{R}^q$ is the restriction of a map in $\mathcal{D}_K^{\Pi_\infty}$.

Also, using the same proof as Lemma A.4, we can assume that $\varphi_k \to \varphi_\infty$ and $\varphi_k^{-1} \to \varphi_\infty^{-1}$ in $C^0_{loc}(\mathbb{R}^2)$, for some homeomorphism $\mathbb{R}^2 \to \mathbb{R}^2$.

By harmonicity, up to subsequences we get $\Psi_k \circ \varphi_k^{-1} \to \Theta_\infty$ in $C^2_{loc}(\varphi_\infty(B_1^2))$, for some $\Theta_\infty : \varphi_\infty(B_1^2) \to \mathbb{R}^q$, so that Θ_∞ is conformal and harmonic.

On the other hand, by Lemma 4.1 Θ_∞ is the harmonic extension of $\gamma \circ \varphi_\infty^{-1}$ and $\Psi_k \to \Theta_\infty \circ \varphi_\infty =: \Psi_\infty$ in $C^0(\overline{B}_1^2)$. By the maximum principle we have $\Pi_\infty^\perp \circ \Theta_\infty = 0$ and thus $\Pi_\infty \circ \Theta_\infty$ is either holomorphic or antiholomorphic on $\varphi_\infty(B_1^2)$ (once Π_∞ is identified with $\mathbb{C}$). Being $\Pi_\infty \circ \Theta_\infty\big|_{\partial\varphi_\infty(B_1^2)} = \Pi_\infty \circ \gamma \circ \varphi_\infty^{-1}$ a Jordan curve, $\Pi_\infty \circ \Theta_\infty$ must be a diffeomorphism from $\varphi_\infty(B_1^2)$ onto its image.

Fix now a compact neighborhood F of $\varphi_\infty(\overline{B}_{1/2}^2)$ in $\varphi_\infty(B_1^2)$, with smooth boundary. Since $\Psi_k \circ \varphi_k^{-1} \to \Theta_\infty$ in $C^1_{loc}(\varphi_\infty(B_1^2))$, we obtain that eventually $\Pi_k \circ \Psi_k \circ \varphi_k^{-1}$ is a diffeomorphism of F onto its image, with

$$\mathrm{dist}(\Pi_k, \Pi_k(x)) < \varepsilon, \qquad x \in F.$$

The fact that eventually $\varphi_k(\overline{B}_{1/2}^2) \subseteq F$ yields the desired contradiction. □

5. Technical iteration lemmas

Definition 5.1. Given $V > 0$ with $V = \lfloor V \rfloor + \frac{1}{2}$, we define $K'(V) := (64V)^2$ and $E'(V) := 2\pi K'(V)^2 D(K'(V))$.

Lemma 5.2. *There exists $0 < \varepsilon_0 < \eta(K)$, depending on $E, V > 0$, $K \geq 1$ and $\mathcal{M}^m$, such that whenever $\Psi \in C^2(\overline{B}^2_r(z), \mathcal{M}^m_{p,\ell})$ is a conformal immersion, critical for the functional* (33) *on $B^2_r(z)$, and Π, Π' are 2-planes satisfying*

- $\Psi(z + r\cdot) \in \mathcal{R}^{\Pi}_{K,\varepsilon_0}$,
- $\frac{1}{2}\int_{B^2_r(z)} |\nabla\Psi|^2 \leq E$,
- $\int_{\Psi^{-1}(B^q_1)} dvol_{g_\Psi} \leq V\pi$,
- $\tau^2 \log(\tau^{-1}) \int_{B^2_r(z)} |A|^4 \, dvol_{g_\Psi} \leq \varepsilon_0$ *for some* $\tau \leq \varepsilon_0$,
- $\operatorname{dist}(\Pi, \Pi') \leq \varepsilon_0$ *and* $\ell \leq \varepsilon_0$,

then the projected multiplicity $N^{\Pi}_{\Psi,z,r}$ satisfies

$$\operatorname{dist}\Big(\fint_{B^{\Pi}_{\eta(K)}} N^{\Pi}_{\Psi,z,s(K)^2}, \mathbb{Z}^+ \Big) < \frac{1}{8}, \tag{51}$$

$$\left| \fint_{B^{\Pi}_{\eta(K)}} N^{\Pi}_{\Psi,z,s(K)^2} - \fint_{B^{\Pi'}_{\eta(K)}} N^{\Pi'}_{\Psi,z,s(K)^2} \right| < \frac{1}{8}, \tag{52}$$

where $\mathbb{Z}^+$ is the set of positive integers.

We remark in passing that vol_{g_Ψ}, i.e. the volume measure induced by Ψ, equals $\frac{1}{2}|\nabla\Psi|^2 \mathcal{L}^2$.

Proof. We can assume $z = 0$ and $r = 1$. Suppose by contradiction that there exists a sequence $\varepsilon_k \downarrow 0$ and planes Π_k, Π'_k making the claim false for $\varepsilon_0 = \varepsilon_k$. Up to subsequences, we can assume that $\Pi_k, \Pi'_k \to \Pi_\infty$, that Ψ_k has a weak limit Ψ_∞ in $W^{1,2}(B^2_1, \mathbb{R}^q)$, with traces $\Psi_\infty\big|_{\partial B^2_s}(s\cdot) = \psi(s\cdot)$ for some $\psi \in \mathcal{D}^{\Pi_\infty}_K$ and all $s \in \{1, s(K), s(K)^2\}$, and that the varifolds $\mathbf{v}_k$ induced by Ψ_k converge to a varifold $\mathbf{v}_\infty$ in $\mathbb{R}^q$.

The arguments used in [12, Section III] and in [11, Section 2] show that Ψ_∞ has a continuous representative on the interior B^2_1, satisfying the convex hull property, namely $\Psi_\infty(\overline{\omega}) \subseteq \operatorname{co}(\Psi_\infty(\partial\omega))$ for all $\omega \subset\subset B^2_1$ (giving in particular $\operatorname{dist}(\Psi_\infty(x), \Psi_\infty(\partial B^2_1)) \geq \frac{1}{2} \geq \frac{1}{4}$ and $B^2_{1/4}(\Psi_\infty(x)) \subseteq B^q_1$ for $x \in \overline{B}^2_{s(K)}$), and that $\mathbf{v}_\infty$ is stationary in

$$U := B^q_1 \setminus \overline{\Psi_\infty(\partial B^2_1)} \supseteq \Psi_\infty(\overline{B}^2_{s(K)}). \tag{53}$$

Let us fix any domain ω such that

$$B^2_{s(K)} \subset\subset \omega \subset\subset \Psi^{-1}_\infty(U), \qquad \operatorname{dist}(\Psi_\infty(x), \partial U) \geq \frac{1}{8} \text{ on } \omega. \tag{54}$$

Since $\|\mathbf{v}_\infty\|(U) \leq V\pi$, by monotonicity we get that the density θ of $\mathbf{v}_\infty$ has

$$\theta(\Psi_\infty(x)) \leq \left(\pi \left(\frac{1}{8} \right)^2 \right)^{-1} \|\mathbf{v}_\infty\| (B^q_{1/8}(\Psi_\infty(x))) \leq 64V \tag{55}$$

for all $x \in \omega$. Hence, setting $K'(V) := (64V)^2$, the aforementioned arguments also give a local parametrized stationary varifold $(\varphi_\infty(\omega), \Theta_\infty, N_\infty \circ \varphi_\infty^{-1})$, where $\Theta_\infty = \Psi_\infty \circ \varphi_\infty^{-1}$ for a suitable $K'(V)$-quasiconformal homeomorphism $\varphi_\infty : \mathbb{R}^2 \to \mathbb{R}^2$ and a suitable $N_\infty \in L^\infty(\omega, \mathbb{Z}^+)$ bounded by $64V$, guaranteeing the Radon measure convergence $\frac{1}{2}|\Psi_k|^2 \mathcal{L}^2 \overset{*}{\rightharpoonup} N_\infty |\partial_1 \Psi_\infty \wedge \partial_2 \Psi_\infty|$.

Notice that there are no bubbling points in ω, since they would provide (nontrivial) compact minimal immersed surfaces without boundary in $\mathbb{R}^q$, which do not exist. Hence, we also get the varifold convergence $\mathbf{v}'_k \overset{*}{\rightharpoonup} \mathbf{v}'_\infty$ and $\mathbf{v}''_k \overset{*}{\rightharpoonup} \mathbf{v}''_\infty$ as $k \to \infty$, as well as the tightness of the sequences $\|\mathbf{v}'_k\|$ and $\|\mathbf{v}''_k\|$, where $\mathbf{v}'_k$ and $\mathbf{v}''_k$ are the varifolds issued by $\Psi_k\big|_{B^2_{s(K)}}$ and $\Psi_k\big|_{B^2_{s(K)^2}}$ respectively, while $\mathbf{v}'_\infty$ and $\mathbf{v}''_\infty$ are the ones issued by $(\varphi_\infty(B^2_{s(K)}), \Theta_\infty, N_\infty \circ \varphi_\infty^{-1})$ and $(\varphi_\infty(B^2_{s(K)^2}), \Theta_\infty, N_\infty \circ \varphi_\infty^{-1})$. The support of $\mathbf{v}''_\infty$ is contained in the plane Π_∞, by the convex hull property enjoyed by Ψ_∞ and the fact that Ψ_∞ maps $\partial B^2_{s(K)^2}$ to Π_∞. Since $\Psi_\infty(\partial B^2_{s(K)^2})$ does not intersect $\Pi_\infty^{-1}(\partial B^{\Pi_\infty}_{\eta(K)})$, the varifold $\mathbf{v}_\infty$ is stationary here and thus, by the constancy theorem, it has a constant density $\nu \in \mathbb{N}$. The area formula then gives

$$\fint_{B^{\Pi_k}_{\eta(K)}} N^{\Pi_k}_{\Psi_k,0,s(K)^2} = \frac{\|(\Pi_k)_* \mathbf{v}''_k\| (B^{\Pi_k}_{\eta(K)})}{\pi \eta(K)^2} \to \frac{\|(\Pi_\infty)_* \mathbf{v}''_\infty\| (B^{\Pi_\infty}_{\eta(K)})}{\pi \eta(K)^2} = \nu.$$

Similarly, $\fint_{B^{\Pi'_k}_{\eta(K)}} N^{\Pi'_k}_{\Psi_k,0,s(K)^2} \to \nu$ as $k \to \infty$. Hence the claim is eventually true, yielding the desired contradiction. □

Remark 5.3. The proof of Lemma 5.2 gives also the following result: whenever $\Psi_k \in C^2(\overline{B}^2_1, \mathcal{M}^m_{p_k,\ell_k})$ is a sequence of conformal immersions such that Ψ_k is critical for the functional (33) (with τ_k, ℓ_k in place of τ, ℓ) and

- $\Psi_k \in \mathcal{R}^{\Pi_k}_{K,\eta(V)}$,
- $\frac{1}{2}\int_{B^2_1} |\nabla \Psi_k|^2 \le E$,
- $\int_{\Psi_k^{-1}(B^q_1)} d\mathrm{vol}_{g_{\Psi_k}} \le V\pi$,
- $\tau_k^2 \log(\tau_k^{-1}) \int_{B^2_1} |A|^4 \, d\mathrm{vol}_{g_{\Psi_k}} \le \varepsilon_k$ for some $\tau_k, \varepsilon_k \to 0$,
- $\ell_k \to 0$,

then up to subsequences $\Psi_k \rightharpoonup \Psi_\infty$ in $W^{1,2}(B^2_1, \mathbb{R}^q)$, with Ψ_∞ continuous and satisfying the convex hull property. Moreover, there exists a $K'(V)$-quasiconformal homeomorphism φ_∞ of $\mathbb{R}^2$ and a multiplicity $N_\infty \in L^\infty(B^2_{s(K)}, \mathbb{Z}^+)$ bounded by $64V$ such that the varifolds induced by $\Psi_k\big|_{B^2_{s(K)}}$ converge in the varifold sense to the local parametrized stationary varifold

$$(\varphi_\infty(B^2_{s(K)}), \Psi_\infty \circ \varphi_\infty^{-1}, N_\infty \circ \varphi_\infty^{-1})$$

and such that the associated mass measures form a tight sequence. This holds more generally if $B^2_{s(K)}$ is replaced with an open subset ω with $\mathcal{L}^2(\partial\omega) = 0$. Finally, we have the convergence of Radon measures $\frac{1}{2}|\nabla \Psi_k|^2 \mathcal{L}^2 \overset{*}{\rightharpoonup} N_\infty |\partial_1 \Psi_\infty \wedge \partial_2 \Psi_\infty| \mathcal{L}^2$.

We now specify δ_0 so that Lemma 4.2 applies, with $\varepsilon := \varepsilon_0$ and $s := s(K)$. Notice that ε_0 and δ_0 still depend on V, K and E.

Lemma 5.4. *Given $E > 0$ and $K \geq 1$ there exists a constant $0 < \varepsilon_0' < \varepsilon_0$ (depending on $E, V, K, \mathcal{M}^m$) with the following property: if a conformal immersion $\Psi \in C^2(\overline{B}_r^2(z), \mathcal{M}_{p,\ell}^m)$ is critical for the functional (33) and satisfies*

- $\Psi(z + r\cdot) \in \mathcal{R}_{K,\delta_0}^{\Pi}$,
- $\frac{1}{2}\int_{B_r^2(z)} |\nabla\Psi|^2 \leq E$,
- $\frac{1}{\pi}\int_{\Psi^{-1}(B_1^q)} dvol_{g_\Psi}, \frac{1}{\pi\eta(K)^2}\int_{\Psi^{-1}(B_{\eta(K)}^q)} dvol_{g_\Psi} \leq V$,
- $\tau^2\log(\tau^{-1})\int_{B_r^2(z)} |A|^4\, dvol_{g_\Psi} \leq \varepsilon_0'$ *for some* $0 < \tau \leq \varepsilon_0'$,
- $0 < \ell \leq \varepsilon_0'$,

then there exist a new point $p' \in \mathcal{M}^m$, new scales r', ℓ' and a new 2-plane Π' with

- $\varepsilon_0' r < r' < s(K)r$,
- $\varepsilon_0' < \ell' < \frac{1}{2}$,
- $\operatorname{dist}(\Pi, \Pi') < \varepsilon_0$,
- $\Psi' := (\ell')^{-1}(\Psi(z + r'\cdot) - p') \in \mathcal{R}_{K'(V),\delta_0}^{\Pi'}$,
- $\frac{1}{2}\int_{B_{r'}^2(z)} |\nabla\Psi'|^2 < E'(V)$,
- $\frac{1}{\pi}\int_{\widetilde{\Psi}^{-1}(B_1^q)} dvol_{g_{\widetilde{\Psi}}}, \frac{1}{\pi\eta(K)^2}\int_{\widetilde{\Psi}^{-1}(B_{\eta(K)}^q)} dvol_{g_{\widetilde{\Psi}}} < \left\lfloor \left(\frac{\eta(K)}{\eta(K)-\varepsilon_0}\right)^2 V \right\rfloor + \frac{1}{2}$.

Proof. We can assume $z = 0$ and $r = 1$. By contradiction, suppose that there is a sequence $\varepsilon_k \downarrow 0$ such that the claim fails (with $\varepsilon_0' = \varepsilon_k$) for all radii $\varepsilon_k < r' < s(K)$, for some Ψ_k and Π_k satisfying all the hypotheses. Up to subsequences, by Remark 5.3, we get a limiting local parametrized stationary varifold $(\Omega_\infty, \Theta_\infty, N_\infty \circ \varphi_\infty^{-1})$ in $\mathbb{R}^q$, where $\Theta_\infty = \Psi_\infty \circ \varphi_\infty^{-1}$ and $\Omega_\infty = \varphi_\infty(B_{s(K)}^2)$ for a suitable $K'(V)$-quasiconformal homeomorphism φ_∞ of the plane. Moreover, assuming also that $\Pi_k \to \Pi_\infty$ and $p_k \to p_\infty$, by weak convergence of traces and Lemma A.4 we still have $\Psi_\infty \in \mathcal{R}_{K,\delta_0}^{\Pi_\infty}$. By the regularity result of [11], Θ_∞ is harmonic. Also, it takes values in the tangent space T at p_∞ (translated to the origin).

Also, by definition of δ_0 and Lemma 4.2, Θ_∞ is a diffeomorphism from $\overline{B}_{1/2}^2$ onto its image and the differential $\nabla\Theta_\infty(0)$ is a conformal linear map of full rank, spanning a plane Π' with $\operatorname{dist}(\Pi_\infty, \Pi') < \varepsilon_0$.

The varifolds $\mathbf{v}_k$ induced by $\Psi_k\big|_{B_{s(K)}^2}$ converge to $\mathbf{v}_\infty$, induced by $(\varphi_\infty(B_{s(K)}^2), \Theta_\infty, N_\infty \circ \varphi_\infty^{-1})$. By the convex hull property enjoyed by Ψ_∞, there exists $y \in B_{s(K)^2}^2$ such that $|\Psi_\infty(y)| \leq \delta_0$. Since $\|\mathbf{v}_\infty\|(B_{\eta(K)}^q) \leq V\pi\eta(K)^2$, the stationarity of $\mathbf{v}_\infty$ near $\Theta_\infty(0)$ implies that its density at $\Psi_\infty(y)$ is at most $\left(\frac{\eta(K)}{\eta(K)-\varepsilon_0}\right)^2 V$. Being $\mathbf{v}_\infty$ stationary in the embedded surface $\Theta_\infty(\varphi_\infty(B_{s(K)}^2))$, the constancy theorem gives that its density θ is a constant integer here.

Thus we have

$$\|\mathbf{v}_\infty\|(\overline{B}_t^q(p_\infty')) < \left(\left\lfloor \left(\frac{\eta(K)}{\eta(K)-\varepsilon_0}\right)^2 V \right\rfloor + \frac{1}{2}\right)\pi t^2, \qquad p_\infty' := \Theta_\infty(0) \in T,$$

for all $t > 0$ small enough. Fix now any $r' < s(K)$ such that we have the strong convergence $\Psi_k(r'\cdot) \to \Psi_\infty(r'\cdot)$ in $C^0(\partial B_1^2 \cup \partial B_{s(K)}^2 \cup \partial B_{s(K)^2}^2)$ along a subsequence. Notice that $\lambda^{-1}\varphi_\infty(r'\cdot) \in \mathcal{D}_{K'(V)}$, where $\lambda := \min_{|x|=r'} |\varphi_\infty(x)|$. Also, the fact that $\Psi_\infty = \Theta_\infty \circ \varphi_\infty$ and the smoothness of Θ_∞ give

$$\big|\Psi_\infty(r'x) - \Psi_\infty(0) - \big\langle \nabla\Theta_\infty(0), \varphi_\infty(r'x)\big\rangle\big| < \frac{\delta_0\,|\nabla\Theta_\infty(0)|}{\sqrt{2}D(K'(V))}\,|\varphi_\infty(x)| \le \delta_0\ell' \tag{56}$$

if r is chosen small enough, where $\ell' := \frac{|\nabla\Theta_\infty(0)|}{\sqrt{2}}\lambda$ and $x \in \overline{B}_1^2$. We can also ensure that

$$\frac{1}{2}\int_{B_{r'}^2} |\nabla\Psi_\infty|^2 \le K'(V)\int_{B^2_{D(K'(V))}} |\nabla\Theta_\infty|^2 < 2K'(V)(\lambda D(K'(V)))^2\pi\,|\nabla\Theta_\infty(0)|^2,$$

as well as, calling $\mathbf{v}'_\infty$ the varifold induced by $(\varphi_\infty(B_{r'}^2), (\ell')^{-1}(\Theta_\infty - p'_\infty), N_\infty \circ \varphi_\infty^{-1})$,

$$\frac{\|\mathbf{v}'_\infty\|\,(\overline{B}_1^q)}{\pi}, \frac{\|\mathbf{v}'_\infty\|\,(\overline{B}^2_{\eta(K)})}{\pi\eta(K)^2} < \Big\lfloor \left(\frac{\eta(K)}{\eta(K)-\varepsilon_0}\right)^2 V\Big\rfloor + \frac{1}{2}.$$

Thanks to (56) and $\lambda^{-1}\varphi_\infty(r\cdot) \in \mathcal{D}_{K'(V)}$, eventually $(\ell')^{-1}(\Psi_k(r\cdot) - p'_k) \in \mathcal{D}^{\Pi'}_{K'(V),\delta_0}$. Moreover, we have

$$\frac{1}{2}\int_{B_{r'}^2(z)} |\nabla\Psi_k|^2 \to \int_{B_{r'}^2(z)} N_\infty\,|\partial_1\Psi_\infty \wedge \partial_2\Psi_\infty| < (\ell')^2E'(V).$$

Also, calling p'_k the closest point to p'_∞ in $\mathcal{M}^m_{p_k,\ell_k}$ (eventually defined and converging to p'_∞, since $\mathcal{M}^m_{p_k,\ell_k} \to T$), from the convergence of the varifolds induced by $(\ell')^{-1}(\Psi_k - p'_k)\big|_{B_{r'}^2}$ to $\mathbf{v}'_\infty$ we get

$$\limsup_{k\to\infty}\frac{\|\mathbf{v}'_k\|\,(B_1^q)}{\pi}, \limsup_{k\to\infty}\frac{\|\mathbf{v}'_k\|\,(B^2_{\eta(K)})}{\pi\eta(K)^2} < \Big\lfloor \left(\frac{\eta(K)}{\eta(K)-\varepsilon_0}\right)^2 V\Big\rfloor + \frac{1}{2}.$$

So eventually $(\ell')^{-1}(\Psi_k(r'\cdot) - p'_k)$ satisfies all the conclusions. This yields the desired contradiction. □

Definition 5.5. Given constants $K'' \ge 1$ and $E'' > 0$, we define $K_0 := \max\{K'(V), K''\}$ and $E_0 := \max\{E'(V), E''\}$. We also let $s_0 := s(K_0)$ and $\eta_0 := \eta(K_0)$.

We fix ε_0 (and thus δ_0) and ε'_0 so that Lemmas 5.2 and 5.4 apply with $K := K_0$, $E := E_0$. Since ε_0 depends on V, we can assume that it is chosen so small that

$$\Big\lfloor \left(\frac{\eta_0}{\eta_0-\varepsilon_0}\right)^2 V\Big\rfloor + \frac{1}{2} = \lfloor V\rfloor + \frac{1}{2} = V. \tag{57}$$

This makes the last conclusion of Lemma 5.4 match one of the hypotheses, making it possible to iterate that result. On the other hand, the constants V, K'', E'' (upon which all the aforementioned constants depend) will be fixed only in Section 6.

Lemma 5.6. *There exists a constant $0 < \varepsilon''_0 < \varepsilon'_0$ with the following property: if a conformal immersion $\Psi \in C^2(\overline{B}_r^2(z), \mathcal{M}^m_{p,\ell})$ satisfies the hypotheses of the previous lemma (with ε''_0*

and K_0 in place of ε_0' and K), then the new point p' and the new radius r' provided by Lemma 5.4 satisfy

$$n^{\Pi,0,\eta_0}_{\Psi,z,s_0^2 r} = n^{\Pi,p',\eta_0\ell'}_{\Psi,z,s_0^2 r'} = n^{\Pi',p',\eta_0\ell'}_{\Psi,z,s_0^2 r'}. \tag{58}$$

Proof. Assume again $z=0$, $r=1$ and, by contradiction, that the first equality in (58) fails, so that we have again two sequences $\varepsilon_k \downarrow 0$ and Ψ_k. We can assume that $\Pi_k \to \Pi_\infty$, $p_k' \to p_\infty'$, $\ell_k' \to \ell_\infty'$ and $r_k' \to r_\infty'$, with $p_\infty' \in \mathcal{M}^m$, $\varepsilon_0' \le \ell_\infty' \le \frac{1}{2}$ and $\varepsilon_0' \le r_\infty' \le s_0$. Moreover, up to further subsequences we get a limiting local parametrized stationary varifold $(\Omega_\infty, \Theta_\infty, N_\infty \circ \varphi_\infty^{-1})$ in $\mathbb{R}^q$. From [11] we know that Θ_∞ is harmonic and N_∞ is constant, so Lemma 4.2 gives that $\Pi_\infty \circ \Theta_\infty$ is a diffeomorphism from $\varphi_\infty(\overline{B}^2_{s_0/2})$ onto its image.

Calling $\mathbf{v}_k$ the varifold issued by $\Psi_k\big|_{B^2_{s_0^2}}$ and $\mathbf{v}_\infty$ the one issued by $(\varphi_\infty(B^2_{s_0^2}), \Theta_\infty, N_\infty \circ \varphi_\infty^{-1})$, we have the varifold convergence $\mathbf{v}_k \overset{*}{\rightharpoonup} \mathbf{v}_\infty$ as $k \to \infty$. The area formula gives

$$\fint_{B^{\Pi_k}_{\eta_0}} N^{\Pi_k}_{\Psi,0,s_0^2} = \frac{\|(\Pi_k)_*\mathbf{v}_k\|\,(B^{\Pi_k}_{\eta_0})}{\pi\eta_0^2} \to \frac{\|(\Pi_\infty)_*\mathbf{v}_\infty\|\,(B^{\Pi_\infty}_{\eta_0})}{\pi\eta_0^2} = N_\infty,$$

since $(\Pi_\infty)_*\mathbf{v}_\infty$ equals an open superset of $B^{\Pi_\infty}_{\eta_0}$ in Π_∞ (by Lemma A.1), equipped with the constant integer multiplicity N_∞. Hence, $n^{\Pi_k,0,\eta_0}_{\Psi_k,0,s_0^2} = N_\infty$ eventually.

Similarly, calling $\mathbf{v}_k$ the varifold induced by $\Psi_k\big|_{B^2_{s_0^2 r_k'}}$ and $\mathbf{v}_\infty$ the varifold induced by $(\varphi_\infty(B^2_{s_0^2 r_\infty'}), \Theta_\infty, N_\infty \circ \varphi_\infty^{-1})$, we have $\mathbf{v}_k' \overset{*}{\rightharpoonup} \mathbf{v}_\infty'$ as $k \to \infty$, as is readily seen by approximating with domains which do not vary along the sequence. Since $(\ell_\infty')^{-1}(\Psi_\infty(r_\infty' \cdot) - p_\infty') \in \mathcal{R}^{\Pi_\infty}_{K_0,\delta_0}$, again $(\Pi_\infty)_*\mathbf{v}_\infty'$ equals a superset of $B^{\Pi_\infty}_{\eta_0\ell_\infty'}$ in Π_∞, with constant density N_∞. This gives again

$$\fint_{B^{\Pi_k}_{\eta_0\ell_k'}(q_k)} N^{\Pi_k}_{\Psi,0,s_0^2 r_k'} = \frac{\|(\Pi_k)_*\mathbf{v}_k'\|\,(B^{\Pi_k}_{\eta_0\ell_k'}(q_k))}{\pi\eta_0^2(\ell_k')^2} \to \frac{\|(\Pi_\infty)_*\mathbf{v}_\infty'\|\,(B^{\Pi_\infty}_{\eta_0\ell_\infty'}(q_\infty))}{\pi\eta_0^2(\ell_\infty')^2(q_\infty)} = N_\infty,$$

where $q_k := \Pi_k(p_k')$ for $k \in \mathbb{N} \cup \{\infty\}$. Hence, $n^{\Pi_k,p_k',\eta_0\ell_k'}_{\Psi_k,0,s_0^2 r_k'} = N_\infty$ eventually. So the first equality in (58) holds eventually, giving the desired contradiction.

The second equality in (58) follows immediately from Lemma 5.2, which gives $n^{\Pi,p',\eta_0\ell'}_{\Psi,z,s_0^2 r'} = n^{\Pi',p',\eta_0\ell}_{\Psi,z,s_0^2 r'}$ since $\operatorname{dist}(\Pi',\Pi) < \varepsilon_0$. □

Lemma 5.7. *Assume that $\Psi \in C^\infty(\overline{B}^2_r(z), \mathcal{M}^m_{p,\ell})$ is a conformal immersion and Π is a 2-plane with $\Psi(z + r\cdot) \in \mathcal{D}^{\Pi}_{K_0,\delta_0}$ and $\frac{1}{2}\int_{B^2_r(z)} |\nabla\Psi|^2 \le E$. If $\int_{B^2_1} |A|^4\, dvol_{g_\Psi}$ and ℓ are sufficiently small, then $\Pi \circ \Psi$ is a diffeomorphism from $\overline{B}^2_{s_0^2}$ onto its image.*

Proof. We can suppose $z = 0$, $r = 1$. Assume by contradiction that the claim does not hold, for a sequence of 2-planes $\Pi_k \to \Pi_\infty$ and immersions $\Psi_k : \overline{B}^2_1 \to \mathcal{M}^m_{p_k,\ell_k}$ with $\ell_k \to 0$ and second fundamental form A_k satisfying

$$\int_{B^2_2} |A_k|^4 \, d\mathrm{vol}_{g_{\Psi_k}} \to 0. \tag{59}$$

Let $\lambda_k \in C^\infty(\overline{B}_1^2)$ be defined by $|\partial_1\Psi_k| = |\partial_2\Psi_k| =: e^{\lambda_k}$ and let $A_{p,\ell}$ and $\widetilde{A}_k$ denote the second fundamental form of $\mathcal{M}^m_{p,\ell} \subseteq \mathbb{R}^q$ and of the immersion Ψ_k in $\mathbb{R}^q$ respectively, so that $\widetilde{A}_k = A_{p_k,\ell_k} + A_k$. Notice that

$$\|A_{p_k,\ell_k}\|_{L^\infty} \le C(\mathcal{M}^m)\ell_k \to 0, \tag{510}$$

so that

$$\int_{B_1^2} \left|\widetilde{A}_k\right|^4 \, d\mathrm{vol}_{g_{\Psi_k}} \to 0. \tag{511}$$

With a slight abuse of notation, let us drop the dependence on k in the subsequent computations. We define the orthonormal frame

$$\widetilde{e}_1 := e^{-\lambda}\partial_1\Psi, \qquad \widetilde{e}_2 := e^{-\lambda}\partial_2\Psi_2 \tag{512}$$

for the tangent space of the immersed surface Ψ. It is straightforward to check that the map $e_1 \wedge e_2 : \overline{B}_1^2 \to \Lambda_2\mathbb{R}^q$ has $|\nabla(e_1\wedge e_2)| = e^\lambda\left|\widetilde{A}\right|$, so

$$\int_{B_1^2} |\nabla(e_1\wedge e_2)|^2 \, d\mathcal{L}^2 = \int_{B_1^2} e^{2\lambda}\left|\widetilde{A}\right|^2 d\mathcal{L}^2 = \int_{B_1^2}\left|\widetilde{A}\right|^2 d\mathrm{vol}_{g_\Psi} \to 0 \tag{513}$$

by Hölder's inequality, since $\int_{B_1^2} d\mathrm{vol}_{g_\Psi} \le c\pi$. We identify the Grassmannian $\mathrm{Gr}_2(\mathbb{R}^q)$ of 2-planes in $\mathbb{R}^q$ with a submanifold of the projectivization of $\Lambda_2\mathbb{R}^q$, by means of Plücker's embedding. For k large enough [3, Lemma 5.1.4] applies and provides a rotated frame (e_1, e_2), given by

$$E := e_1 + ie_2 = e^{i\theta}\widetilde{E}, \qquad \widetilde{E} := \widetilde{e}_1 + i\widetilde{e}_2, \tag{514}$$

for a suitable real function $\theta \in W^{1,2}(B_1^2)$ minimizing $\int_{B_1^2} |\nabla\theta + \widetilde{e}_1 \cdot \nabla\widetilde{e}_2|^2$ (in particular, θ and E are smooth functions on $\overline{B}_1^2$) and with $\|\nabla E\|_{L^2}^2$ becoming arbitrarily small as $k \to \infty$. We will assume in the sequel that $\|\nabla E\|_{L^2}^2 \le 1$. Observe that, whenever $\alpha, \beta \in C^1(\overline{B}_1^2)$,

$$\begin{aligned}\partial_1\alpha\partial_2\beta - \partial_2\alpha\partial_1\beta &= \frac{1}{4}(\partial_1\alpha + \partial_2\beta)^2 + \frac{1}{4}(\partial_2\alpha - \partial_1\beta)^2 - \frac{1}{4}(\partial_1\alpha - \partial_2\beta)^2 - \frac{1}{4}(\partial_2\alpha + \partial_1\beta)^2 \\ &= |\partial_z(\alpha + i\beta)|^2 - |\partial_{\overline{z}}(\alpha + i\beta)|^2 .\end{aligned}$$

Hence, being $\widetilde{e}_1 + i\widetilde{e}_2 = 2e^{-\lambda}\partial_{\overline{z}}\Psi$ and $\partial_z\Psi\cdot\partial_z\Psi = \partial_{\overline{z}}\Psi\cdot\partial_{\overline{z}}\Psi = 0$ by conformality, we get

$$\begin{aligned}-(\partial_1\widetilde{e}_1\cdot\partial_2\widetilde{e}_2 - \partial_2\widetilde{e}_1\cdot\partial_1\widetilde{e}_2) &= 4\left|\partial_{\overline{z}}(e^{-\lambda}\partial_{\overline{z}}\Psi)\right|^2 - 4\left|\partial_z(e^{-\lambda}\partial_{\overline{z}}\Psi)\right|^2 \\ &= 4\partial_{\overline{z}}(e^{-\lambda}\partial_{\overline{z}}\Psi)\cdot\partial_z(e^{-\lambda}\partial_z\Psi) - 4\partial_{\overline{z}}(e^{-\lambda}\partial_z\Psi)\cdot\partial_z(e^{-\lambda}\partial_{\overline{z}}\Psi) \\ &= 4e^{-2\lambda}(\partial^2_{\overline{z}\overline{z}}\Psi\cdot\partial^2_{zz}\Psi - \partial^2_{\overline{z}z}\Psi\cdot\partial^2_{\overline{z}z}\Psi - \partial_{\overline{z}}\lambda\partial_{\overline{z}}\Psi\cdot\partial_{zz}\Psi - \partial_z\lambda\partial_z\Psi\cdot\partial_{\overline{z}\overline{z}}\Psi) \\ &\quad + 2e^{-2\lambda}\partial_{\overline{z}}\lambda\partial_{\overline{z}}(\partial_z\Psi\cdot\partial_z\Psi) + 2e^{-\lambda}\partial_z\lambda\partial_z(\partial_{\overline{z}}\Psi\cdot\partial_{\overline{z}}\Psi) \\ &= 4e^{-2\lambda}(\partial^2_{\overline{z}\overline{z}}\Psi\cdot\partial^2_{zz}\Psi - \partial^2_{\overline{z}z}\Psi\cdot\partial^2_{\overline{z}z}\Psi - \partial_{\overline{z}}\lambda\partial_{\overline{z}}\Psi\cdot\partial_{zz}\Psi - \partial_z\lambda\partial_z\Psi\cdot\partial_{\overline{z}\overline{z}}\Psi).\end{aligned}$$

On the other hand we have

$$2e^{2\lambda}\partial_z\lambda = \partial_z(e^{2\lambda}) = \partial_z(2\partial_{\overline{z}}\Psi\cdot\partial_z\Psi) = \partial_{\overline{z}}(\partial_z\Psi\cdot\partial_z\Psi) + 2\partial_{\overline{z}}\Psi\cdot\partial^2_{zz}\Psi = 2\partial_{\overline{z}}\Psi\cdot\partial_{zz}\Psi,$$

$$\begin{aligned}\Delta(e^{2\lambda}) &= 4\partial^2_{\overline{z}z}(2\partial_{\overline{z}}\Psi\cdot\partial_z\Psi) = 8\partial_{\overline{z}}(\partial_{\overline{z}}\Psi\cdot\partial^2_{zz}\Psi) + 4\partial_{\overline{zz}}(\partial_z\Psi\cdot\partial_z\Psi)\\ &= 8(\partial_{\overline{zz}}\Psi\cdot\partial_{zz}\Psi - \partial_{\overline{z}z}\Psi\cdot\partial_{\overline{z}z}\Psi) + 4\partial_{zz}(\partial_{\overline{z}}\Psi\cdot\partial_{\overline{z}}\Psi)\\ &= 8(\partial_{\overline{zz}}\Psi\cdot\partial_{zz}\Psi - \partial_{\overline{z}z}\Psi\cdot\partial_{\overline{z}z}\Psi),\end{aligned}$$

so we arrive at

$$\partial_1\widetilde{e}_1\cdot\partial_2\widetilde{e}_2 - \partial_2\widetilde{e}_1\cdot\partial_1\widetilde{e}_2 = -\frac{\Delta(e^{2\lambda})}{2e^{2\lambda}} + 8\partial_{\overline{z}}\lambda\partial_z\lambda = -\Delta\lambda. \tag{515}$$

Alternatively, since the projections of $\partial_j\widetilde{e}_1$ and $\partial_k\widetilde{e}_2$ onto the tangent space of the immersion Ψ are orthogonal (being the projection of $\partial_j\widetilde{e}_1$ a multiple of $\widetilde{e}_2$ and the projection of $\partial_k\widetilde{e}_2$ a multiple of $\widetilde{e}_1$),

$$\partial_1\widetilde{e}_1\cdot\partial_2\widetilde{e}_2 - \partial_2\widetilde{e}_1\cdot\partial_1\widetilde{e}_2 = e^{2\lambda}(\widetilde{A}(\widetilde{e}_1,\widetilde{e}_1)\cdot\widetilde{A}(\widetilde{e}_2,\widetilde{e}_2) - \widetilde{A}(\widetilde{e}_1,\widetilde{e}_2)\cdot\widetilde{A}(\widetilde{e}_1,\widetilde{e}_2)) = e^{2\lambda}K,$$

by Gauss' formula, K denoting the Gaussian curvature of the immersed surface. But, by the well-known formula for the curvature of a conformal metric, we have $K = -e^{-2\lambda}\Delta\lambda$, which gives again (515). Moreover,

$$\begin{aligned}\partial_1 e_1\cdot\partial_2 e_2 - \partial_2 e_1\cdot\partial_1 e_2 &= \Im\left\langle\nabla\overline{E};\nabla E\right\rangle = \Im\left\langle\nabla\overline{\widetilde{E}} - i\overline{\widetilde{E}}\otimes\nabla\theta;\nabla\widetilde{E} + i\widetilde{E}\otimes\nabla\theta\right\rangle\\ &= \Im\left\langle\nabla\overline{\widetilde{E}};\nabla\widetilde{E}\right\rangle = \partial_1\widetilde{e}_1\cdot\partial_2\widetilde{e}_2 - \partial_2\widetilde{e}_1\cdot\partial_1\widetilde{e}_2,\end{aligned}$$

since $\left\langle\overline{\widetilde{E}}\otimes\nabla\theta;\widetilde{E}\otimes\nabla\theta\right\rangle$ is real and $\left\langle -i\overline{\widetilde{E}}\otimes\nabla\theta;\nabla\widetilde{E}\right\rangle = \overline{\left\langle\nabla\overline{\widetilde{E}};i\widetilde{E}\otimes\nabla\theta\right\rangle}$. Thus, calling $\mu\in C^\infty(\overline{B}^2_1)$ the solution to

$$\begin{cases}-\Delta\mu = \partial_1 e_1\cdot\partial_2 e_2 - \partial_2 e_1\cdot\partial_1 e_2 & \text{on } B^2_1\\ \mu = 0 & \text{on } \partial B^2_1,\end{cases}$$

we obtain that $\lambda-\mu$ is harmonic and, by Wente's inequality,

$$\|\mu\|_{L^\infty} \le C(q)\left(\|\nabla e_1\|^2_{L^2} + \|\nabla e_2\|^2_{L^2}\right) \le C(q). \tag{516}$$

Since $\lambda < e^{2\lambda}$, for all $x\in\overline{B}^2_{3/4}$ we get

$$(\lambda-\mu)(x) = \fint_{B^2_{1/4}(x)}(\lambda-\mu) \le \fint_{B^2_{1/4}(x)} e^{2\lambda} + \|\mu\|_{L^\infty} \le \frac{E}{\mathcal{L}^2(B^2_{1/4})} + C(q). \tag{517}$$

Together with (516), this gives an upper bound for λ on $B^2_{3/4}$, depending only on V, q. Although this is sufficient for the present purposes, one can also get a lower bound for λ on $B^2_{s_0}$. Indeed, calling M the right-hand side of (517), we obtain that $M-(\lambda-\mu)$ is a nonnegative harmonic function on $B^2_{3/4}$. Moreover, the length of the curve $\Psi\big|_{\partial B^2_{s_0}}$ is

$$\int_{\partial B^2_{s_0}} e^\lambda \ge 2\pi\eta_0 \tag{518}$$

by the area formula, since the composition of $\Psi\big|_{\partial B^2_{s_0}}$ with the radial projection onto $\partial B^2_{\eta_0}$ (which does not increase the length) is surjective (being a generator of the fundamental group of $\partial B^2_{\eta_0}$). Hence, there exists some $x \in \partial B^2_{s_0}$ such that $\lambda(x) \geq \log\left(s_0^{-1}\eta_0\right)$. We deduce that

$$\inf_{B^2_{s_0}}(M - (\lambda - \mu)) \leq M + C(q) - \log(s_0^{-1}\eta_0) \tag{519}$$

and so, by Harnack's inequality, the supremum of $M - (\lambda - \mu)$ on $B^2_{s_0}$ is bounded by a constant depending only on V, s_0, η_0, q. This, together with (517) and (516), gives

$$\|\lambda\|_{L^\infty(B^2_{s_0})} \leq C(V, E, \eta_0, q). \tag{520}$$

The mean curvature of the immersion Ψ is $\widetilde{H} = \frac{1}{2e^{2\lambda}}(\widetilde{A}(\partial_1\Psi, \partial_1\Psi) + \widetilde{A}(\partial_2\Psi, \partial_2\Psi)) = -\frac{\Delta\Psi}{2e^\lambda}$ (notice that $\Delta\Psi$ is already orthogonal to the tangent space of the immersion, since $\partial_z\Psi \cdot \Delta\Psi = 4\partial_z\Psi \cdot \partial^2_{\bar{z}z}\Psi = 2\partial_{\bar{z}}(\partial_z\Psi \cdot \partial_z\Psi) = 0$). So we get

$$\int_{B^2_{3/4}} |\Delta\Psi_k|^4 \, d\mathcal{L}^2 = 16\int_{B^2_{3/4}} \left|\widetilde{H}_k\right|^4 e^{2\lambda_k} \, d\mathrm{vol}_{g_{\Psi_k}} \leq C(c, q)\int_{B^2_{3/4}} \left|\widetilde{A}_k\right|^4 d\mathrm{vol}_{g_{\Psi_k}} \to 0. \tag{521}$$

Since $s_0 \leq \frac{1}{2}$, this implies that (Ψ_k) is a bounded sequence in $W^{2,4}(B^2_{s_0})$ (see Lemma A.2 applied to $\Psi_k(\frac{3}{4}\cdot)$), so by the compact embedding $W^{2,4}(B^2_{s_0}) \hookrightarrow C^1(\overline{B}^2_{s_0})$ we obtain a strong limit Ψ_∞ in $C^1(\overline{B}^2_{s_0})$, up to subsequences. Thus Ψ_∞ is weakly conformal and, by (521), it is also harmonic. Lemma 4.2 applies (with $\Psi = \Psi_\infty(s_0\cdot)$ and $\varphi = \mathrm{id}_{\mathbb{R}^2}$) and gives that $\Pi_\infty \circ \Psi_\infty$ is a diffeomorphism from $\overline{B}^2_{s_0/2} \supseteq \overline{B}^2_{s_0^2}$ onto its image, hence the same is eventually true for $\Pi_k \circ \Psi_k$, giving the desired contradiction. ☐

6. Multiplicity one in the limit

Theorem 6.1. *Assume $\Phi \in C^\infty(\overline{B}^2_r(z), \mathcal{M}^m)$ is a conformal immersion, critical for* (31) *on $B^2_r(z)$ and satisfying*

- $\sigma^2\log(\sigma^{-1})\int_{B^2_s}|A|^4 \, dvol_{g_\Phi} \leq \frac{\varepsilon_0''}{E_0}\int_{B^2_s} dvol_{g_\Phi}$ *for all* $0 < s \leq r$,
- $\frac{1}{2}\int_{B^2_1}|\nabla\Phi|^2 \leq \min\{V\pi, E_0\}$,
- $\ell^{-1}(\Phi(z + r\cdot) - \Psi(z)) \in \mathcal{R}^\Pi_{K_0,\delta_0}$ *for some* $\ell \geq \sqrt{\sigma/\varepsilon_0''}$.

Then, if σ and ℓ are small enough (independently of each other), we have $n^{\Pi,\Phi(z),\eta_0\ell}_{\Phi,z,s_0^2 r} = 1$.

Proof. Let $r_0 := r$, $p_0 := \Phi(z)$, $\ell_0 := \ell$, $\tau_0 := \sigma\ell_0^{-2}$ and $\Pi_0 := \Pi$. Notice that

$$\Psi_0 := \ell^{-1}(\Phi - \Phi(z)) = \ell_0^{-1}(\Phi - p_0)$$

is critical for (33), with $\tau := \tau_0 \leq \varepsilon_0''$. Thus Lemma 5.4 applies (if ℓ is small enough), giving a new radius $\varepsilon_0' r_0 < r_1 < s_0 r_0$, a new point $p' \in \mathcal{M}^m$, a new scale ℓ' and a new 2-plane Π'. Setting $r_1 := r'$, $p_1 := p_0 + \ell_0 p'$, $\ell_1 := \ell'\ell_0$, $\tau_1 := \sigma\ell_1^{-2}$, $\Pi_1 := \Pi'$ and recalling (57), the map

$$\Psi_1 := (\ell')^{-1}(\Psi_0 - p') = \ell_1^{-1}(\Psi - p_1)$$

still satisfies the hypotheses of Lemma 5.4, except possibly for $\tau_1 \le \varepsilon_0'$, with the parameters r_1, τ_1, p_1, ℓ_1: indeed, notice that (assuming $\tau_1 < 1$)

$$\tau_1^2 \log(\tau_1^{-1}) \int_{B^2_{r_1}(z)} |A|^4 \, d\mathrm{vol}_{g_{\Psi_1}} \le \tau_1^2 \log(\sigma^{-1}) \int_{B^2_{r_1}(z)} |A|^4 \, d\mathrm{vol}_{g_{\Psi_1}}$$
$$= \ell_1^{-2}\sigma^2 \log(\sigma^{-1}) \int_{B^2_{r_1}(z)} |A|^4 \, d\mathrm{vol}_{g_\Phi} \le \frac{\varepsilon_0''\ell_1^{-2}}{E_0} \int_{B^2_{r_1}(z)} d\mathrm{vol}_{g_\Phi} = \frac{\varepsilon_0''}{2E_0} \int_{B^2_{r_1}(z)} |\nabla \Psi_1|^2 \le \varepsilon_0''.$$

Hence, we can iterate and define $r_j, p_j, \ell_j, \tau_j, \Pi_j$, for $j = 0, 1, \dots,$ up to a maximum index $k \ge 1$ for which the constraint $\tau_k \le \varepsilon_0'$ is no longer verified: such k exists since $\tau_j \ge 2^j \tau_0$. This implies

$$\int_{B^2_{r_k}(z)} |A|^4 \, d\mathrm{vol}_{g_{\Psi_k}} \le \frac{\varepsilon_0''}{\tau_k^2 \log(\sigma^{-1})} \le \frac{\varepsilon_0''}{(\varepsilon_0')^2 \log(\sigma^{-1})}.$$

If σ and ℓ are small enough, Lemma 5.7 applies and, together with Lemma A.1, gives $n^{\Pi_k, p_k, \eta_0 \ell_k}_{\Psi_k, z, s_0^2 r_k} = 1$. Also, Lemma 5.6 applies for all $j = 0, \dots, k-1$, giving

$$n^{\Pi, \Phi(z), \eta_0 \ell}_{\Phi, z, s_0^2 r} = n^{\Pi_0, p_0, \eta_0 \ell_0}_{\Psi_0, z, s_0^2 r_0} = n^{\Pi_1, p_1, \eta_0 \ell_1}_{\Psi_1, z, s_0^2 r_1} = \dots = n^{\Pi_k, p_k, \eta_0 \ell_k}_{\Psi_k, z, s_0^2 r_k} = 1. \qquad \square$$

As in Section 3, assume now that $\Phi_k : \Sigma \to \mathcal{M}^m$ is a sequence of critical points for

$$\int_\Sigma d\mathrm{vol}_{g_{\Phi_k}} + \sigma_k^2 \int_\Sigma (1 + |A|^2)^2 \, d\mathrm{vol}_{g_{\Phi_k}} \tag{61}$$

with controlled area, namely

$$\lambda \le \int_\Sigma d\mathrm{vol}_{g_{\Phi_k}} \le \Lambda,$$

and with

$$\sigma_k \to 0, \qquad \sigma_k^2 \log(\sigma_k^{-1}) \int_\Sigma (1 + |A|^2)^2 \, d\mathrm{vol}_{g_{\Phi_k}} \to 0.$$

By the main result of [12], up to subsequences the varifolds $\mathbf{v}_k$ induced by Φ_k converge to a parametrized stationary varifold.

In the remainder of the paper, we will assume for simplicity that there is no bubbling and no degeneration of the conformal structure, so that the limiting varifold $\mathbf{v}_\infty$ is induced by a weak limit $\Phi_\infty \in W^{1,2}(\Sigma, \mathcal{M}^m)$ of Φ_k, with a multiplicity N_∞. The arguments will apply also to the general case, working on suitable domains different from Σ.

Assuming without loss of generality that the conformal classes induced by Φ_k converge, we fix a metric on Σ inducing the limiting conformal class. The limiting parametrized stationary varifold has the form $(\Sigma_\infty, \Theta_\infty, N_\infty)$, where $\Theta_\infty : \Sigma_\infty \to \mathcal{M}^m$ is a smooth branched minimal immersion and $\varphi_\infty : \Sigma \to \Sigma_\infty$ is (locally) a quasiconformal homeomorphism such that $\Psi_\infty = \Theta_\infty \circ \varphi_\infty$.

By the regularity result in [11], which was already exploited in Section 5, N_∞ is locally a.e. constant and thus a.e. constant (being Σ connected).

Definition 6.2. We set $\mu := \inf_k \mathcal{H}^2_\infty(\Phi_k(\Sigma))$, where we recall that, for a set $S \subseteq \mathbb{R}^q$,

$$\mathcal{H}^2_\infty(S) := \inf\left\{ \sum_j \pi \operatorname{diam}(E_j)^2 \mid S \subseteq \bigcup_j E_j \right\}.$$

Lemma 6.3. *We have $\mu > 0$.*

Proof. Fix any Lebesgue point x_0 for Φ_∞ and $d\Phi_\infty$, such that $d\Phi_\infty(x_0)$ has full rank. Working in a conformal chart centered at x_0, there exists a radius such that $\Phi_\infty(r\cdot)\big|_{\partial B_1^2}$ has a $W^{1,2}$ representative, $\Phi_k(r\cdot) \to \Phi_\infty(r\cdot)$ in $C^0(\partial B_1^2)$ (up to subsequences) and

$$\|\Phi_\infty(r\cdot) - \Phi_\infty(0) - \langle \nabla\Phi_\infty(0), r\cdot\rangle\|_{L^\infty(\partial B_1^2)} < \frac{1}{2}\min_{x\in\partial B_1^2} |\langle \nabla\Phi_\infty(0), ry\rangle| . \tag{62}$$

By Lemma A.1, calling $\Pi \subseteq \mathbb{R}^q$ the 2-plane spanned by $\nabla\Phi_\infty$ and $p_\infty := \Pi \circ \Phi_\infty(0) \in \Pi$, eventually we have

$$B_s^\Pi(p_\infty) \subseteq \Pi \circ \Phi_k(B_r^2), \qquad s := \frac{1}{2}\min_{x\in B_1^2} |\langle \nabla\Phi_\infty(0), ry\rangle| . \tag{63}$$

But $\mathcal{H}^2_\infty(B_s^\Pi(p_\infty)) = \pi s^2$, since on 2-planes $\mathcal{H}^2_\infty$ equals the standard 2-dimensional Lebesgue measure. Thus

$$\pi s^2 \le \mathcal{H}^2_\infty(\Pi \circ \Phi_k(\Sigma)) \le \mathcal{H}^2_\infty(\Phi_k(\Sigma)). \tag{64}$$

Since the argument can be repeated starting from an arbitrary subsequence, the claim is established. □

Definition 6.4. We let $T_{K''}$ denote the set of bad points z which are not Lebesgue for $d\Phi_\infty$, or such that $d\Phi_\infty(z)$ does not have full rank, or such that

$$\max_{|x|=1} |\langle \nabla\Phi_\infty(0), x\rangle| > K'' \min_{|x|=1} |\langle \nabla\Phi_\infty(0), x\rangle| \tag{65}$$

in conformal coordinates centered at z. By (66) we have $\nu_\infty(T_{K''}) \to 0$ as $K'' \to \infty$: we now specify the value of $K'' \ge 1$ in such a way that $\nu_\infty(T_{K''}) \le \frac{\mu}{4}$. We also set $E'' := 4\pi \|N_\infty\|_{L^\infty} ((K'')^2 + 1)$. Notice that now also the constants K_0, E_0, s_0, η_0, as well as ε_0, δ_0, ε_0' and ε_0'', are determined.

Lemma 6.5. *There exists $V > 0$ such that, calling S_k the set of points $z \in \Sigma$ satisfying*

- $\int_{\Phi_k^{-1}(B_\ell^q(\Phi_k(z)))} dvol_{g_{\Phi_k}} < V\pi\ell^2$ *for all* $0 < \ell < 1$,
- $\sigma_k^2 \log(\sigma_k^{-1}) \int_{B_r^2(z)} |A|^4 \, dvol_{g_{\Phi_k}} < \varepsilon_0'' \int_{B_r^2(z)} dvol_{g_{\Phi_k}}$ *for all* $0 < r < 1$,

we have $\int_{S_k} dvol_{g_{\Phi_k}} \ge \frac{\mu}{2}$ *for all k large enough (depending on ε) and* $V = \lfloor V \rfloor + \frac{1}{2}$.

Proof. Let $\mathcal{B}_k$ be the Borel set of points $p \in \Phi_k(\Sigma)$ such that $\|\mathbf{v}_k\|(B_\ell^q(p)) > V\pi\ell^2$ for some radius $0 < \ell < 1$. By Besicovitch's covering lemma, we can find a finite or countable collection of points $p_i \in \mathcal{B}_k$ and radii ℓ_i such that

$$\|\mathbf{v}_k\|(B_{\ell_i}^q(p_i)) \ge V\pi\ell_i^2, \qquad \mathbf{1}_{\mathcal{B}_k} \le \sum_i \mathbf{1}_{B_{\ell_i}^q(p_i)} \le \mathfrak{N}$$

for some universal $\mathfrak{N}$ depending only on q. Thus,

$$\mathcal{H}^2_\infty(\mathcal{B}_k) \le \sum_i \pi \ell_i^2 \le V^{-1} \sum_i \|\mathbf{v}_k\| \, (B^q_{\ell_i}(p_i)) \le V^{-1}\mathfrak{N}\Lambda.$$

Choosing $V := \left\lceil \frac{4\mathfrak{N}\Lambda}{\mu} \right\rceil + \frac{1}{2}$ (i.e. $V := \min\left\{n \in \mathbb{N} : n \ge \frac{4\mathfrak{N}\Lambda}{\mu}\right\} + \frac{1}{2}$), we get

$$\|\mathbf{v}_k\| \, (\mathcal{M}^m \setminus \mathcal{B}_k) \ge \mathcal{H}^2(\Phi_k(\Sigma) \setminus \mathcal{B}_k) \ge \mathcal{H}^2_\infty(\Phi_k(\Sigma) \setminus \mathcal{B}_k) \ge \mu - \mathcal{H}^2_\infty(\mathcal{B}_k) \ge \frac{3}{4}\mu.$$

Similarly, calling $\mathcal{B}'_k$ be the Borel set of points z such that the second condition fails for some radius $0 < r < 1$, we get a collection of points $z_i \in \mathcal{B}'$ and radii r_i such that

$$\sigma_k^2 \log(\sigma_k^{-1}) \int_{B^2_{r_i}(z_i)} |A|^4 \, d\mathrm{vol}_{g_{\Phi_k}} \ge \varepsilon''_0 \int_{B^2_{r_i}(z_i)} d\mathrm{vol}_{g_{\Phi_k}}, \qquad \mathbf{1}_{\mathcal{B}'_k} \le \sum_i \mathbf{1}_{B^2_{r_i}(z_i)} \le \mathfrak{N}.$$

Thus we get

$$\begin{aligned} \mathrm{vol}_{g_{\Phi_k}}(\mathcal{B}'_k) &\le \sum_i \mathrm{vol}_{g_{\Phi_k}}(B^2_{r_i}(z_i)) \le (\varepsilon''_0)^{-1} \sigma_k^2 \log(\sigma_k^{-1}) \sum_i \int_{B^2_{r_i}(z_i)} |A|^4 \, d\mathrm{vol}_{g_{\Phi_k}} \\ &\le (\varepsilon''_0)^{-1} \mathfrak{N} \sigma_k^2 \log(\sigma_k^{-1}) \int_\Sigma |A|^4 \, d\mathrm{vol}_{g_{\Phi_k}} \to 0. \end{aligned}$$

Hence, for k so large that $\mathrm{vol}_{g_{\Phi_k}}(\mathcal{B}'_k) \le \frac{\mu}{4}$, we get

$$\mathrm{vol}_{g_{\Phi_k}}(\mathcal{B}'_k)(\Sigma \setminus (\Phi_k^{-1}(\mathcal{B}_k) \cup \mathcal{B}'_k)) \ge \mathrm{vol}_{g_{\Phi_k}}(\Phi_k^{-1}(\mathcal{M}^m \setminus \mathcal{B}_k)) - \mathrm{vol}_{g_{\Phi_k}}(\mathcal{B}'_k) \ge \frac{3}{4}\mu - \frac{\mu}{4} \ge \frac{\mu}{2},$$

as $\|\mathbf{v}_k\| = (\Phi_k)_* \mathrm{vol}_{g_{\Phi_k}}$. The claim follows by taking $S_k := \Sigma \setminus (\Phi_k^{-1}(\mathcal{B}_k) \cup \mathcal{B}'_k)$. □

Theorem 6.6. *We have $N_\infty = 1$.*

Proof. Up to subsequences, we can assume that $\overline{S}_k$ converges in the Hausdorff topology to some compact set S_∞. Setting $\nu_k := \mathrm{vol}_{g_{\Phi_k}}$, by [12] we know that (up to further subsequences) $\Phi_k \rightharpoonup \Phi_\infty$ in $W^{1,2}(\Sigma)$ and $\nu_k \overset{*}{\rightharpoonup} \nu_\infty$, for suitable Φ_∞ and ν_∞ satisfying, in local conformal coordinates for Σ,

$$\nu_\infty = N_\infty \, |\partial_1 \Phi_\infty \wedge \partial_2 \Phi_\infty| \, . \tag{66}$$

We remark that $\nu_\infty(S_\infty) \ge \frac{\mu}{2}$: indeed, for any compact neighborhood F of S_∞, we have $S_k \subseteq F$ eventually and so

$$\nu_\infty(F) \ge \limsup_{k\to\infty} \nu_k(F) \ge \limsup_{k\to\infty} \nu_k(S_k) \ge \frac{\mu}{2}. \tag{67}$$

We now show that $N_\infty = 1$ on $S_\infty \setminus T_{K''}$: fix any $z \in S_\infty \setminus T_{K''}$ and choose conformal coordinates centered at z. We can find points $z_k \in S_k$ such that $z_k \to 0$ and conformal reparametrizations $\widetilde{\Psi}_k$ of $\Phi_k(z_k + \cdot)$, by means of diffeomorphisms converging smoothly to the identity. By weak convergence $\widetilde{\Phi}_k \rightharpoonup \Phi_\infty$ in $W^{1,2}$, we can find an arbitrarily small radius r such that

$$\widetilde{\Phi}_k(r\cdot) \to \Phi_\infty(r\cdot) \quad \text{in } C^0(\partial B^2_1 \cup \partial B^2_{s_0} \cup \partial B^2_{s_0^2}) \tag{68}$$

up to further subsequences, as well as

$$|\Phi_\infty(rx) - \Phi_\infty(0) - \langle \nabla\Phi_\infty(0), rx\rangle| < \delta_0 \ell\, |x| \quad \text{for } x \in \partial B_1^2 \cup \partial B_{s_0}^2 \cup \partial B_{s_0^2}^2, \tag{69}$$

$$\frac{1}{2}\int_{B_r^2} |\nabla\Phi_\infty|^2 \le (2r)^2\pi\, |\nabla\Phi_\infty(0)|^2 \le 4\ell^2\pi((K'')^2+1), \tag{610}$$

with $\ell := r\min_{|x|=1}|\langle\nabla\Phi_\infty(0), x\rangle|$. Thanks to the definition of E'' and (66), eventually $\Psi_k := \ell^{-1}(\widetilde{\Phi}_k - \Phi_\infty(0))$ satisfies the hypotheses of Lemma 6.1, provided that r (and thus ℓ) is small enough. We infer that $n^{\Pi,0,\eta_0}_{\Psi_k,0,s_0^2} = 1$, where Π is the 2-plane spanned by $\nabla\Phi_\infty(0)$.

Since r can be chosen arbitrarily small (possibly changing the subsequence guaranteeing (68)), the argument used in the proof of [12, Lemma III.10] shows that $N_\infty(z) = 1$. Thus $N_\infty = 1$ on $S_\infty \setminus T_{K''}$, which has positive Lebesgue measure (being $\nu_\infty(S_\infty \setminus T_{K''}) \ge \frac{\mu}{4} > 0$). Since N_∞ is a.e. constant, we have $N_\infty = 1$ a.e. Alternatively, $n^{\Pi,0,\eta_0}_{\Psi_k,0,s_0^2} = 1$ gives

$$\left|\frac{\|\Pi_*\mathbf{v}'_k\|\,(B^\Pi_{\eta_0})}{\pi\eta_0^2} - 1\right| < \frac{1}{8},$$

where $\mathbf{v}'_k$ is induced by $\Psi_k\big|_{B^2_{s_0^2}}$. Assuming without loss of generality that $\nabla\Theta_\infty(\varphi_\infty(0)) \ne 0$, the convergence of $\mathbf{v}_k$ to the varifold $\mathbf{v}'_\infty$ induced by $(\varphi_\infty(B^2_{s_0^2 r}), \Theta_\infty, N_\infty)$ and the injectivity of $\Pi\circ\Theta_\infty$ on $B^2_{s_0^2 r}$ (which holds provided that r is small enough and that the chain rule $d\Psi_\infty(0) = d\Theta_\infty(\varphi_\infty(0))\circ d\varphi_\infty(0)$ applies) give

$$\frac{\|\Pi_*\mathbf{v}'_k\|\,(B^\Pi_{\eta_0})}{\pi\eta_0^2} \to \frac{\|\Pi_*\mathbf{v}'_\infty\|\,(B^\Pi_{\eta_0})}{\pi\eta_0^2} = N_\infty,$$

so again we conclude that $N_\infty = 1$ a.e. □

Appendix.

Lemma A.1. *Assume that* $F \in C^0(\overline{B}_1^2, \mathbb{R}^2)$ *satisfies*

$$|F(x) - \varphi(x)| \le \delta \qquad \text{for all } x \in \partial B_1^2 \tag{A1}$$

for some $0 < \delta < 1$ *and some homeomorphism* $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$, *with* $\varphi(0) = 0$ *and* $\min_{|x|=1}|\varphi(x)| = 1$. *Then*

$$F(B_1^2) \supseteq B_{1-\delta}^2. \tag{A2}$$

Proof. It suffices to show that, for a fixed $y \in B^2_{1-\delta}$, the closed curve $\Gamma' := F\big|_{\partial B_1^2}$ is not contractible in $\mathbb{R}^2\setminus\{y\}$: if we had $y \notin F(B_1^2)$, i.e. $y \notin F(\overline{B}_1^2)$, then F would provide a homotopy from Γ' to the constant curve $F(0)$ in $\mathbb{R}^2\setminus\{y\}$, yielding a contradiction.

Let $\Gamma := \varphi\big|_{\partial B_1^2}$ and $\gamma := \Gamma' - \Gamma$, we have $|\gamma(x)| \le \delta$ for all $x \in \partial B_1^2$. Hence, Γ is homotopic to Γ' in $\mathbb{R}^2 \setminus B^2_{1-\delta} \subseteq \mathbb{R}^2\setminus\{y\}$ by means of the homotopy

$$\Gamma + t\gamma, \qquad 0 \le t \le 1.$$

So we are left to show that Γ is not contractible in $\mathbb{R}^2 \setminus \{y\}$, i.e. that $\Gamma - y$ is not contractible in $\mathbb{R}^2 \setminus \{0\}$. The curve $\Gamma - y$ is homotopic to Γ in $\mathbb{R}^2 \setminus \{0\}$, by means of the homotopy

$$\Gamma - ty, \qquad 0 \le t \le 1,$$

which avoids the origin since $|y| < 1$. Finally, Γ is not contractible in $\mathbb{R}^2 \setminus \{0\}$, since φ (once restricted to a homeomorphism of $\mathbb{R}^2 \setminus \{0\}$) induces an automorphism of $\pi_1(\mathbb{R}^2 \setminus \{0\})$ sending the class of the generator $\mathrm{id}_{\partial B_1^2}$ to the class of Γ. Hence, $\Gamma - y$ is not contractible in $\mathbb{R}^2 \setminus \{0\}$, too, as desired. □

Lemma A.2. *For a function $\Psi \in C^\infty(\overline{B}_1)$ and a $0 < \tau < 1$ we have*

$$\|\Psi\|_{W^{2,4}(B_\tau^2)} \le C(\tau)(\|\Delta\Psi\|_{L^4(B_1^2)} + \|\nabla\Psi\|_{L^2(B_1^2)} + \|\Psi\|_{L^2(B_1^2)}).$$

Proof. Given two radii $0 < r < s \le 1$, let us choose a cut-off function $\rho \in C_c^\infty(B_s^2)$ with $\rho = 1$ on B_r^2. Since $\rho\Psi \in C_c^\infty(\mathbb{R}^2)$, standard Calderón–Zygmund estimates give

$$\begin{aligned}\left\|\nabla^2\Psi\right\|_{L^p(B_r^2)} &\le \left\|\nabla^2(\rho\Psi)\right\|_{L^p(\mathbb{R}^2)} \le C(p)\,\|\Delta(\rho\Psi)\|_{L^p(\mathbb{R}^2)} \\ &\le C(p,r,s)(\|\Delta\Psi\|_{L^p(B_s^2)} + \|\nabla\Psi\|_{L^p(B_s^2)} + \|\Psi\|_{L^p(B_s^2)}).\end{aligned} \tag{A3}$$

Setting $t := \frac{1+\tau}{2}$ and applying (A3) with $p := 2$, $r := t$ and $s := 1$ we get

$$\left\|\nabla^2\Psi\right\|_{L^2(B_t^2)} \le C(\tau)(\|\Delta\Psi\|_{L^2(B_1^2)} + \|\nabla\Psi\|_{L^2(B_1^2)} + \|\Psi\|_{L^2(B_1^2)}),$$

hence $\|\Psi\|_{W^{2,2}(B_t^2)}$ is bounded by the desired quantity. Using Sobolev's embedding $W^{2,2}(B_t^2) \hookrightarrow W^{1,4}(B_t^2)$ and (A3) with $p := 4$, $r := \tau$ and $s := t$, we obtain

$$\begin{aligned}\|\Psi\|_{W^{2,4}(B_\tau^2)} &\le C(\|\Delta\Psi\|_{L^4(B_t^2)} + \|\Psi\|_{W^{2,2}(B_t^2)}) \\ &\le C(\|\Delta\Psi\|_{L^4(B_1^2)} + \|\nabla\Psi\|_{L^2(B_1^2)} + \|\Psi\|_{L^2(B_1^2)}).\end{aligned}$$ □

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Lemma A.3. *Given a sequence $\psi_k : \mathbb{C} \to \mathbb{C}$ of K-quasiconformal homeomorphisms with the normalization conditions*

$$\psi_k(0) = 0, \qquad \psi_k(1) = 1,$$

there exists a K-quasiconformal homeomorphism $\psi_\infty : \mathbb{C} \to \mathbb{C}$ satisfying the same normalization condition and such that, up to subsequences, $\psi_k \to \psi_\infty$ and $\psi_k^{-1} \to \psi_\infty^{-1}$ in $C^0_{loc}(\mathbb{C})$.

Proof. Let $\mu_k \in \mathcal{E}_K$ be defined by $\partial_z\psi_k = \mu_k\partial_{\overline{z}}\psi_k$. Existence and uniqueness of a K-quasiconformal homeomorphism satisfying this equation and the normalization conditions is shown in [4, Theorem 4.30].

Given $M > 0$, we consider the set $\mathcal{E}_K^M := \left\{\mu \in \mathcal{E}_K : \mu = 0 \text{ a.e. on } \mathbb{C} \setminus B_M^2\right\}$. If F^μ denotes the normal solution to the equation $\partial_{\overline{z}}F^\mu = \mu\partial_z F^\mu$ (in the sense of [4, Theorem 4.24]), then F^μ satisfies estimates (4.21) and (4.24) in [4]. Applying them with $z_1 := 1$, $z_2 := 0$, we infer that also the map $f^\mu := F^\mu(1)^{-1}F^\mu$ satisfies estimates of the form

$$|f^\mu(z_1) - f^\mu(z_2)| \le C\,|z_1 - z_2|^\alpha + C\,|z_1 - z_2|\,, \tag{A4}$$

$$|z_1 - z_2| \le C\,|f^\mu(z_1) - f^\mu(z_2)|^\alpha + C\,|f^\mu(z_1) - f^\mu(z_2)|\,, \tag{A5}$$

with C and α depending only on K and M. Given a sequence of homeomorphisms $f_k : \mathbb{C} \to \mathbb{C}$ satisfying these estimates, Ascoli–Arzelà theorem applies to f_k and f_k^{-1} and so we can extract a subsequence (not relabeled) such that

$$f_k \to f_\infty, \quad f_k^{-1} \to \widetilde{f}_\infty \quad \text{in } C^0_{loc}(\mathbb{C}).$$

From $f_k^{-1} \circ f_k = f_k \circ f_k^{-1} = \mathrm{id}_{\mathbb{C}}$ we get $\widetilde{f}_\infty \circ f_\infty = f_\infty \circ \widetilde{f}_\infty = \mathrm{id}_{\mathbb{C}}$ and thus $f_\infty : \mathbb{C} \to \mathbb{C}$ is a homeomorphism, with $\widetilde{f}_\infty = f_\infty^{-1}$. Also, since $f_k(z), f_k^{-1}(z) \to \infty$ uniformly as $z \to \infty$, we deduce that the canonical extensions $\widehat{f}_k : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ converge uniformly to $\widehat{f}_\infty$ and that the same holds for $\widehat{f}_k^{-1}$.

We now closely examine the proof of [4, Theorem 4.30]: let $\widetilde{\mu}_k \in \mathcal{E}^1_K$ be given by equation (4.25) in [4], with $\mu_k \mathbf{1}_{\mathbb{C}\setminus B^2_1}$ in place of μ, and

$$g_k : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}, \qquad g_k(z) := \widehat{f^{\widetilde{\mu}_k}}(z^{-1})^{-1}.$$

This map corresponds to the map f^{μ_1} in the aforementioned proof (with μ_k in place of μ). The lower bound (A5), applied with $f^{\widetilde{\mu}_k}$ and $z_1 := f^{\widetilde{\mu}_k}(z^{-1})$, $z_2 := 0$, shows that $|f_k(z)|$ is bounded above by some M, for all k and all $z \in \overline{B}^2_1$. Hence, defining $\mu_{k,2}$ as in equation (4.27) in [4] (with μ_k in place of μ), we get $\mu_{k,2} \in \mathcal{E}^M_{\widetilde{K}}$ for some $\widetilde{K} \ge 1$. Calling $h_k : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ the associated quasiconformal homeomorphism, normalized so that $h_k(0) = 0$ and $h_k(1) = 1$, by the above argument we obtain the uniform convergence

$$g_k \to g_\infty, \quad g_k^{-1} \to g_\infty, \quad h_k \to h_\infty, \quad h_k^{-1} \to h_\infty$$

up to subsequences, for suitable homeomorphisms g_∞ and h_∞ of the Riemann sphere $\widehat{\mathbb{C}}$. Setting $\psi_\infty := h_\infty \circ g_\infty\big|_{\mathbb{C}}$ and observing that $\psi_k = h_k \circ g_k\big|_{\mathbb{C}}$, we get the desired convergence $\psi_k \to \psi_\infty$ and $\psi_k^{-1} \to \psi_\infty^{-1}$ in $C^0_{loc}(\mathbb{C})$.

Finally, we show that ψ_∞ is a K-quasiconformal homeomorphism. Given an open rectangle $R \subset\subset \mathbb{C}$, [4, Lemma 4.12] gives

$$\mathcal{L}^2(\psi_k(R)) = \int_R (|\partial_z \psi_k|^2 - |\partial_{\overline{z}} \psi_k|^2) \ge \int_R (1 - k^2)\,|\partial_z \psi_k|^2 \ge (1 - k^2)k^2 \int_R |\partial_{\overline{z}} \psi_k|^2\,,$$

where $k := \frac{K-1}{K+1}$. Since $\mathcal{L}^2(\psi_k(R)) \to \mathcal{L}^2(\psi_\infty(R))$ we deduce that ψ_k is bounded in $W^{1,2}(R)$, thus ψ_∞ is the limit of ψ_k in the weak $W^{1,2}_{loc}(\mathbb{C})$-topology. Given $\rho, \psi^1, \psi^2 \in C^\infty_c(\mathbb{C})$, integration by parts shows that

$$\int \rho(\partial_1\psi^1\partial_2\psi^2 - \partial_2\psi^1\partial_1\psi^2) = -\int (\partial_1\rho\psi^1\partial_2\psi^2 - \partial_2\rho\psi^1\partial_1\psi^2). \tag{A6}$$

Writing $\psi_k = \varphi_k^1 + i\psi_k^2$, a standard density argument shows that (A6) still holds with ψ^1, ψ^2 replaced by ψ_k^1, ψ_k^2, for $k \in \mathbb{N} \cup \{\infty\}$. Hence, observing that $|\partial_z \psi_k|^2 - |\partial_{\bar{z}} \psi_k|^2 = (\partial_1 \psi_k^1 \partial_2 \psi_k^2 - \partial_2 \psi_k^1 \partial_1 \psi_k^2)$, we get

$$\int \rho(|\partial_z \psi_k|^2 - |\partial_{\bar{z}} \psi_k|^2) \to \int \rho(|\partial_z \psi_k|^2 - |\partial_{\bar{z}} \psi_k|^2). \tag{A7}$$

Defining the positive measures $\nu_k := (|\partial_z \psi_k|^2 - |\partial_{\bar{z}} \psi_k|^2)\mathcal{L}^2$, up to further subsequences we can assume that $\nu_k \overset{*}{\rightharpoonup} \nu_\infty$ as Radon measures. For any rectangle R such that $\nu_\infty(\partial R) = 0$, approximating $\mathbf{1}_R$ from above and below with smooth functions and applying A7 we get

$$\int_R (|\partial_z \psi_k|^2 - |\partial_{\bar{z}} \psi_k|^2) \to \int_R (|\partial_z \psi_k|^2 - |\partial_{\bar{z}} \psi_k|^2).$$

By monotonicity of the left-hand side, this actually holds for every rectangle R. On the other hand, by lower semicontinuity of the L^2-norm,

$$\begin{aligned}\int_R (1-k^2)\, |\partial_z \psi_\infty|^2 &\le \liminf_{k\to\infty} \int_R (1-k^2)\, |\partial_z \psi_k|^2 \le \lim_{k\to\infty} (|\partial_z \psi_k|^2 - |\partial_{\bar{z}} \psi_k|^2) \\ &= \int_R (|\partial_z \psi_\infty|^2 - |\partial_{\bar{z}} \psi_\infty|^2).\end{aligned}$$

Since R is arbitrary, we get $|\partial_{\bar{z}} \psi_\infty| \le k\, |\partial_z \psi_\infty|$ a.e., as desired. □

Lemma A.4. *Given a sequence $\varphi_k \in \mathcal{D}_K$, there exists $\varphi_\infty \in \mathcal{D}_K$ such that, up to subsequences, $\varphi_k \to \varphi_\infty$ and $\varphi_k^{-1} \to \varphi_\infty^{-1}$ in $C^0_{loc}(\mathbb{C})$.*

Proof. Let $\mu_k \in \mathcal{E}_K$ be defined by $\partial_z \varphi_k = \mu_k \partial_{\bar{z}} \varphi_k$ for all k and let $\psi_k : \mathbb{C} \to \mathbb{C}$ be the unique K-quasiconformal homeomorphism satisfying the same differential equation, as well as $\psi_k(0) = 0$, $\psi_k(1) = 1$ (see [4, Theorem 4.30]).

By Lemma A.3, up to subsequences there exists a K-quasiconformal homeomorphism ψ_∞ such that $\psi_k \to \psi_\infty$ and $\psi_k^{-1} \to \psi_\infty^{-1}$ in $C^0_{loc}(\mathbb{C})$.

The map $\psi_k \circ \varphi_k^{-1} : \mathbb{C} \to \mathbb{C}$ is a biholomorphism and fixes the origin, so it equals the multiplication by a nonzero complex number λ_k, i.e. $\psi_k = \lambda_k \varphi_k$. On the other hand,

$$|\lambda_k| = \min_{x \in \partial B_1^2} |\psi_k(x)| \to \min_{x \in \partial B_1^2} |\psi_\infty(x)| \in (0, \infty).$$

Hence, up to further subsequences we can suppose that $\lambda_k \to \lambda_\infty \in \mathbb{C} \setminus \{0\}$. The statement follows with $\varphi_\infty := \lambda_\infty^{-1} \psi_\infty$. □

Remark A.5. In general, given $\varphi_k \in \mathcal{D}_K$ (for $k \in \mathbb{N} \cup \{\infty\}$) with $\varphi_k \to \varphi_\infty$ and $\varphi_k^{-1} \to \varphi_\infty^{-1}$ locally uniformly, it is not true that the corresponding Beltrami coefficients satify $\mu_k \overset{*}{\rightharpoonup} \mu_\infty$ in $L^\infty(\mathbb{C})$. For instance, let $\mu_0(z) := \frac{1}{2}$ if $\Re(z) \in \bigcup_{n\in\mathbb{Z}} \left[n, n+\frac{1}{2}\right)$ and $\mu_0(z) := -\frac{1}{2}$ otherwise. Then the bi-Lipschitz homeomorphism $\psi_0 : \mathbb{C} \to \mathbb{C}$ given by

$$\psi_0(x+iy) := \begin{cases} n + \frac{9}{5}(x-n) + \frac{3}{5}iy = n + \frac{6}{5}(z-n) + \frac{3}{5}(\bar{z}-n) & n \le x \le n + \frac{1}{2} \\ n + \frac{4}{5} + \frac{x-n}{5} + \frac{3}{5}iy = n + \frac{4}{5} + \frac{2}{5}(z-n) - \frac{1}{5}(\bar{z}-n) & n + \frac{1}{2} \le x \le n+1 \end{cases}$$

satisfies $\partial_{\overline{z}}\psi_0 = \mu_0 \partial_z \psi_0$, with the normalization $\psi_0(0)$ and $\psi_0(1) = 1$. So $\mu_k := \mu_0(2^k \cdot)$ and $\psi_k := 2^{-k}\psi_0(2^k \cdot)$ satisfy $\partial_{\overline{z}}\psi_k = \mu_k \partial_z \psi_k$ with the same normalization. Moreover, they converge uniformly to $\psi_\infty(x+iy) = x + \frac{3}{5}iy = \frac{4}{5}z + \frac{1}{5}\overline{z}$, together with their inverses. The homeomorphism ψ_∞ satisfies $\partial_{\overline{z}}\psi_\infty = \mu_\infty \partial_z \psi_\infty$ with $\mu_\infty := \frac{1}{4}$, but $\mu_k \overset{*}{\rightharpoonup} 0$. Dividing each ψ_k by $\min_{|z|=1} |\psi_k(z)|$, we obtain a counterexample in the class $\mathcal{D}_{1/2}$.

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