Danilov's Theorem

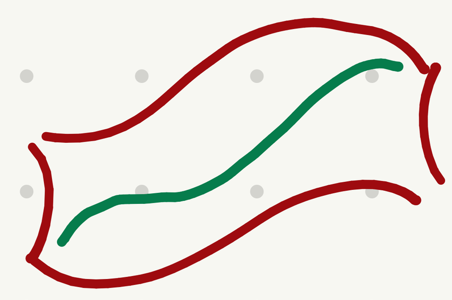
Chapter 1: Rational Equivalence and Chow Ring

Let X be a smooth n-variety. For $k=0,\dots,n$ we define

$$Z_k(X) := \mathbb{Z}\{Y \text{ k-subvariety of } X\} \qquad \text{k-CYCLES of } X.$$

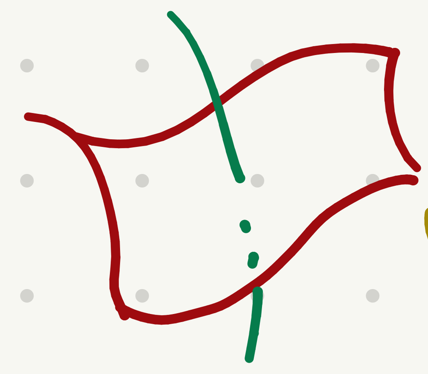
We are interested in studying the intersection of these cycles:

Problem: what's the "codimension" of $Y_1 \cap Y_2$? NO



We need a concept of "TRASVERSALITY".

To achieve this, we introduce RATIONAL EQUIVALENCE:



YES

let $f: X \to \mathbb{P}^1_{\mathbb{C}}$ be a rational function. We can associate a divisor to f in the following sense:

$$\operatorname{div}(f) := \sum_{\substack{V \subseteq X \\ (n-1)\text{-subvariety}}} \operatorname{ord}_V(f)[V] \in Z_{n-1}(X).$$

▸ EXAMPLE $\mathbb{P}^1_{\mathbb{C}}$

Let $X = \mathbb{P}^1_{\mathbb{C}}$ and with coordinates $[z_0 : z_1]$, take the meromorphic function

$$f: \mathbb{C} = U_0 \longrightarrow \mathbb{C} = U_0 \qquad t \longmapsto \frac{t-\alpha}{t-\beta}$$

and extend it to get for $t = \frac{x_1}{x_0}$:

$$\tilde{f}: \mathbb{P}^1_{\mathbb{C}} \longrightarrow \mathbb{P}^1_{\mathbb{C}} \qquad [z_0 : z_1] \longmapsto [z_1 - \beta z_0 : z_1 - \alpha z_0]$$

and then $\operatorname{div}(f) = [1:\alpha] - [1:\beta]$.

Doing this also in higher dimensions let us define:

$$\operatorname{Rat}_k(X) := \mathbb{Z}\{\operatorname{div}(f) \mid W \subseteq X \ (k+1)\text{-subvar}, \ f: W \to \mathbb{P}^1_{\mathbb{C}} \text{ rational}\}$$

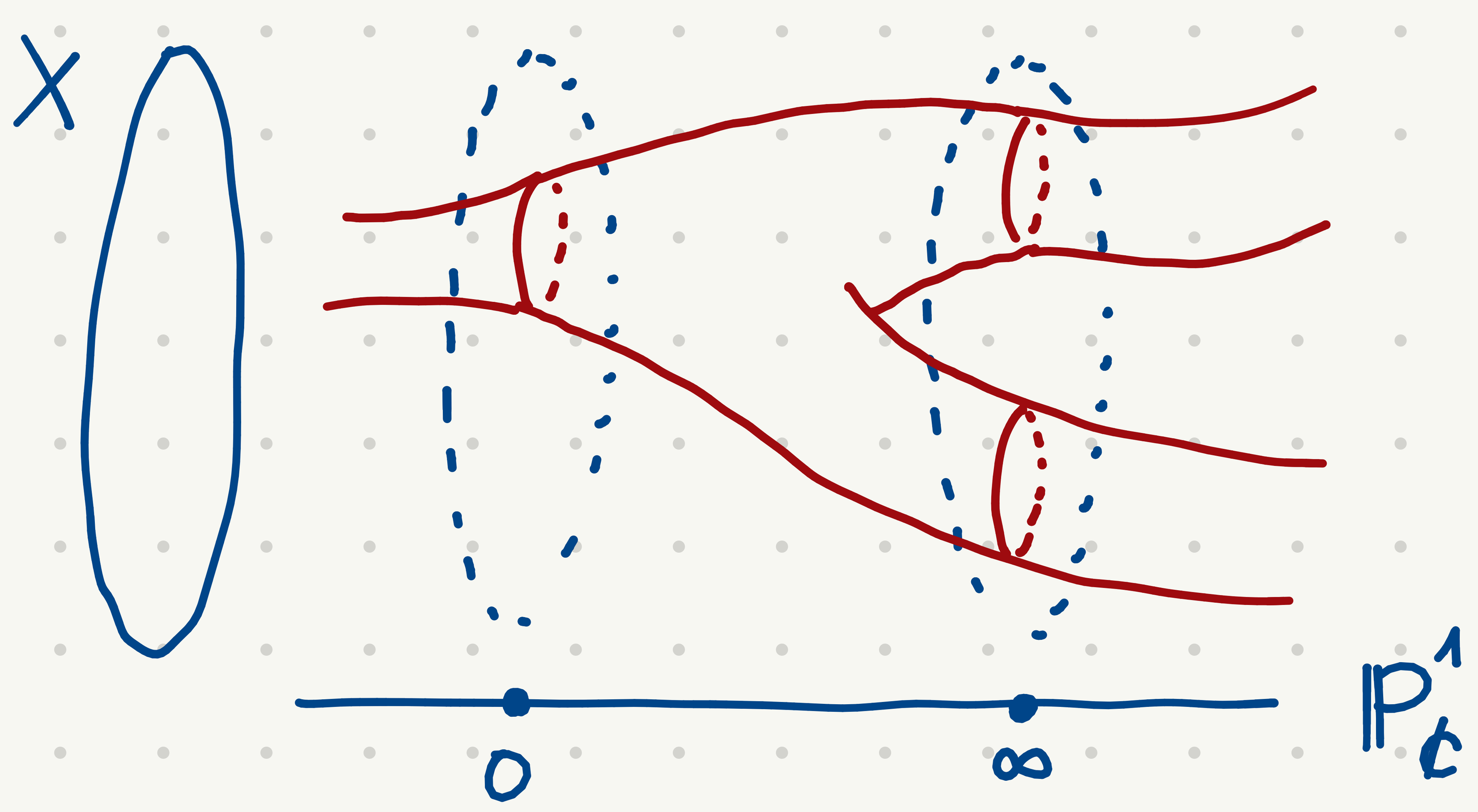
Therefore the CHOW GROUP is defined as

$$A_*(X) = \bigoplus A_k(X) \quad \text{where} \quad A_k(X) = Z_k(X) / \operatorname{Rat}_k(X).$$

▶ EXAMPLE $\mathbb{P}^1_{\mathbb{C}}$

Taking from the previous example, $[P],[Q] \in Z_k(\mathbb{P}^1_{\mathbb{C}})$, if $P=[1:\alpha]$, $Q=[1:\beta]$ then $[P]=[Q]$ in $A_0(\mathbb{P}^1_{\mathbb{C}})$.

For who, like me, is novel to this notion, the book "3264 & all that" by Eisenbud-Harris gives a different visual idea:



Φ subvariety of $X \times \mathbb{P}^1_{\mathbb{C}}$ s.t. $\Phi \notin X \times \{t\}$ $\forall t \in \mathbb{P}^1_{\mathbb{C}}$. Then $Rat(X)$ generated by $[\Phi \cap (X \times \{t_0\})] - [\Phi \cap (X \times \{t_\infty\})]$.

This is all done because of the following...

▶ MOVING LEMMA

Let X be a smooth quasi-projective variety. Then

(*) for every $[U],[V] \in A_*(X)$, $\exists\ U' \in [U]$, $V' \in [V]$ such that U' and V' meet trasversely.

(*) the element $[U \cap V]$ does not depend on representatives.

This let us define a PRODUCT on $A_*(X)$. Let

$$A^k(X) := A_{n-k}(X) \quad \text{and} \quad A^*(X) = \bigoplus_{k=0}^{n} A^k(X).$$

Then letting $[U]\cdot[V] := [U \cap V]$ induces by linearity a well defined product.

$\Rightarrow$ $A^*(X)$ is a GRADED RING.

We also now want to take in consideration the following result, which gives a FUNDAMENTAL CLASS for $A^*(X)$...

THEOREM

Let X be irreducible of dimension n. Then

$$A_n(X) \cong \mathbb{Z} \cong \langle [X] \rangle.$$

EXAMPLE Computing $A^*(\mathbb{P}^1_{\mathbb{C}})$

By the Theorem, we know $A^0(\mathbb{P}^1_{\mathbb{C}}) = \langle [\mathbb{P}^1_{\mathbb{C}}] \rangle$.

On the other hand, $Z_0(\mathbb{P}^1_{\mathbb{C}}) = \mathbb{Z}\{P \in \mathbb{P}^1_{\mathbb{C}}\}$ and by the previous example $P \sim Q$ $\forall\, P, Q \in \mathbb{P}^1_{\mathbb{C}}$. Hence

$$A^1(\mathbb{P}^1_{\mathbb{C}}) = A_0(\mathbb{P}^1_{\mathbb{C}}) = \langle [P] \rangle \simeq \mathbb{Z}.$$

$\boxed{\text{Chapter 2}}$: Generators for $A^*(X_\Sigma)$

Let Σ be a rational fan in $N_{\mathbb{R}}$ lattice (i.e. fan in $N_{\mathbb{Q}}$).

We have constructed X_Σ:

$$\sigma \in \Sigma \text{ in } N_{\mathbb{R}} \Rightarrow \sigma^\vee \text{ in } M_{\mathbb{R}} = N_{\mathbb{R}}^\vee \Rightarrow S_\sigma := \sigma^\vee \cap M.$$

Now we get $U_\sigma := \mathrm{Specm}(\mathbb{C}[S_\sigma])$.

"points of U_σ" $\leftrightarrow$ "$\gamma : S_\sigma \to \mathbb{C}$ semigroup homomorphism"

$$p \equiv m_p \longleftrightarrow \quad m \longmapsto \chi^m(p) = p_1^{m_1} \cdot \ldots \cdot p_n^{m_n}$$

DEF. Distinguished Point

We define for $\sigma \in \Sigma$ the element $\gamma_\sigma \in U_\sigma \subseteq X_\Sigma$:

$$\gamma_\sigma : S_\sigma \longrightarrow \mathbb{C}$$
$$m \longmapsto \begin{cases} 1 & \text{if } m \in \sigma^\perp \cap M \\ 0 & \text{elsewhere} \end{cases}.$$

> **IDEA:** γ_σ is the "0-element" in the affine space $U_\sigma \subseteq \mathbb{C}^{\dim \sigma}$, and elsewhere is non-trivial.

Recall that $T = U_{\{0\}} = \mathrm{Specm}(\mathbb{C}[\chi^{\pm e_1^\vee}, \dots, \chi^{\pm e_n^\vee}]) \cong (\mathbb{C}^*)^n$.

This gives an action: $\quad (T \ni \gamma : (z_1, \dots, z_n) \to z_1 \dots z_n \in \mathbb{C})$

$$U_\sigma \times T \longrightarrow U_\sigma = \mathrm{SpecM}(\mathbb{C}[\chi^{m_1}, \dots, \chi^{m_\ell}])$$

$$(\gamma, (z_1, \dots, z_n)) \longmapsto (z_1, \dots, z_n) \cdot \gamma \qquad .$$

We define the ORBIT CLOSURE of σ:

$$\mathcal{O}(\sigma) := T \cdot \gamma_\sigma \subseteq U_\sigma \subseteq X_\Sigma .$$

Now, we can define the fundamental subvarieties we'll use:

$\forall\, \sigma \in \Sigma$ take $\quad F_\sigma := \overline{\mathcal{O}(\sigma)}^{\,\mathrm{zar}}$.

These subvarieties have the following properties:

- $\mathrm{codim}(F_\sigma) = \dim(\sigma)$.
- by construction of γ_σ and U_τ invariance of T we get: $\quad F_\sigma \cap U_\tau \neq \emptyset \iff \overline{\tau} \supseteq \sigma$.
- by construction $F_{\{0\}} = T$.

THEOREM Orbit-cone Correspondence

Let X_Σ be a toric variety. Then we have a correspondence:

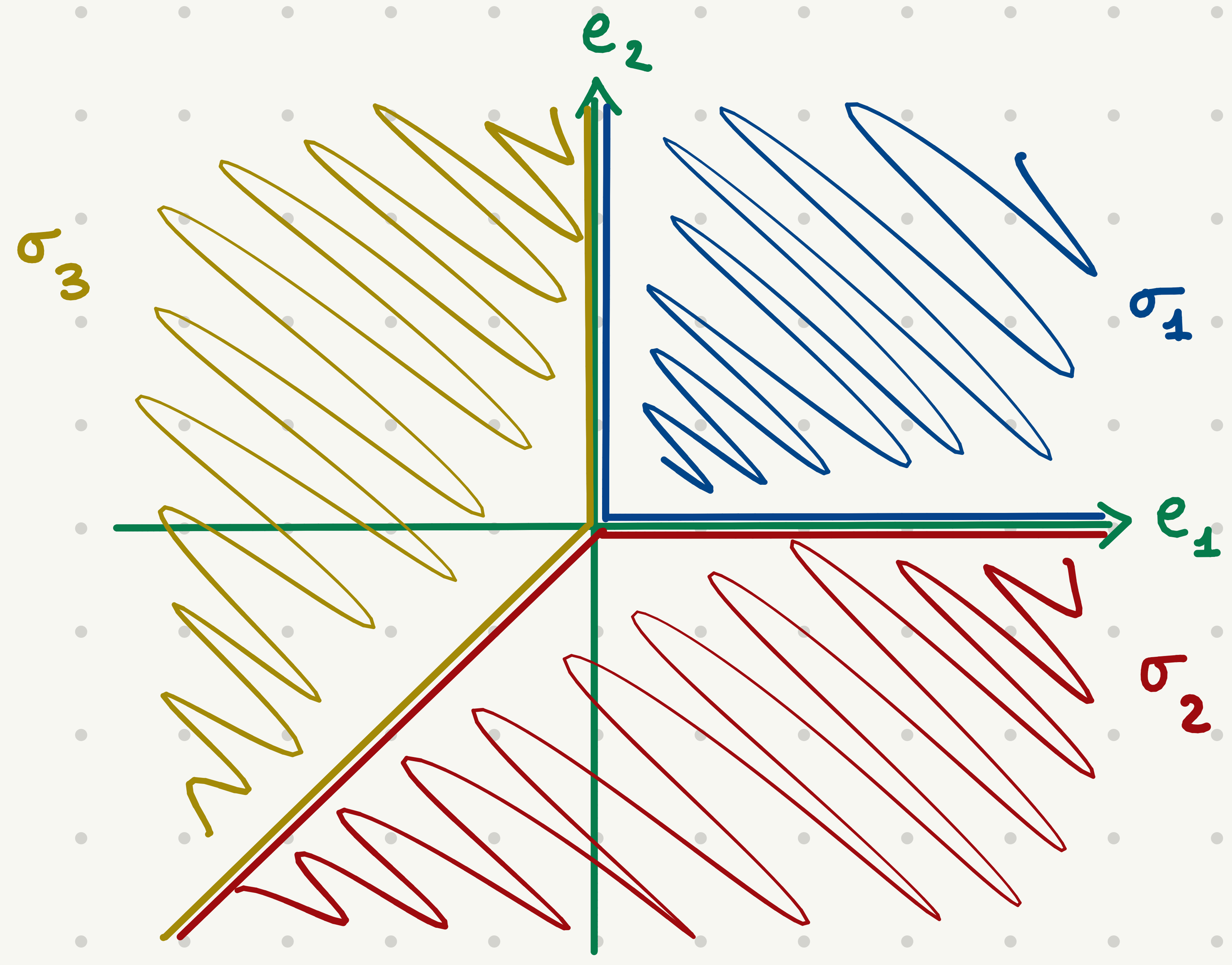
$$\{\text{cones of } \Sigma\} \overset{1:1}{\longleftrightarrow} \{T\text{-orbits in } X_\Sigma\}$$

$$\sigma \longleftrightarrow \mathcal{O}(\sigma) \qquad .$$

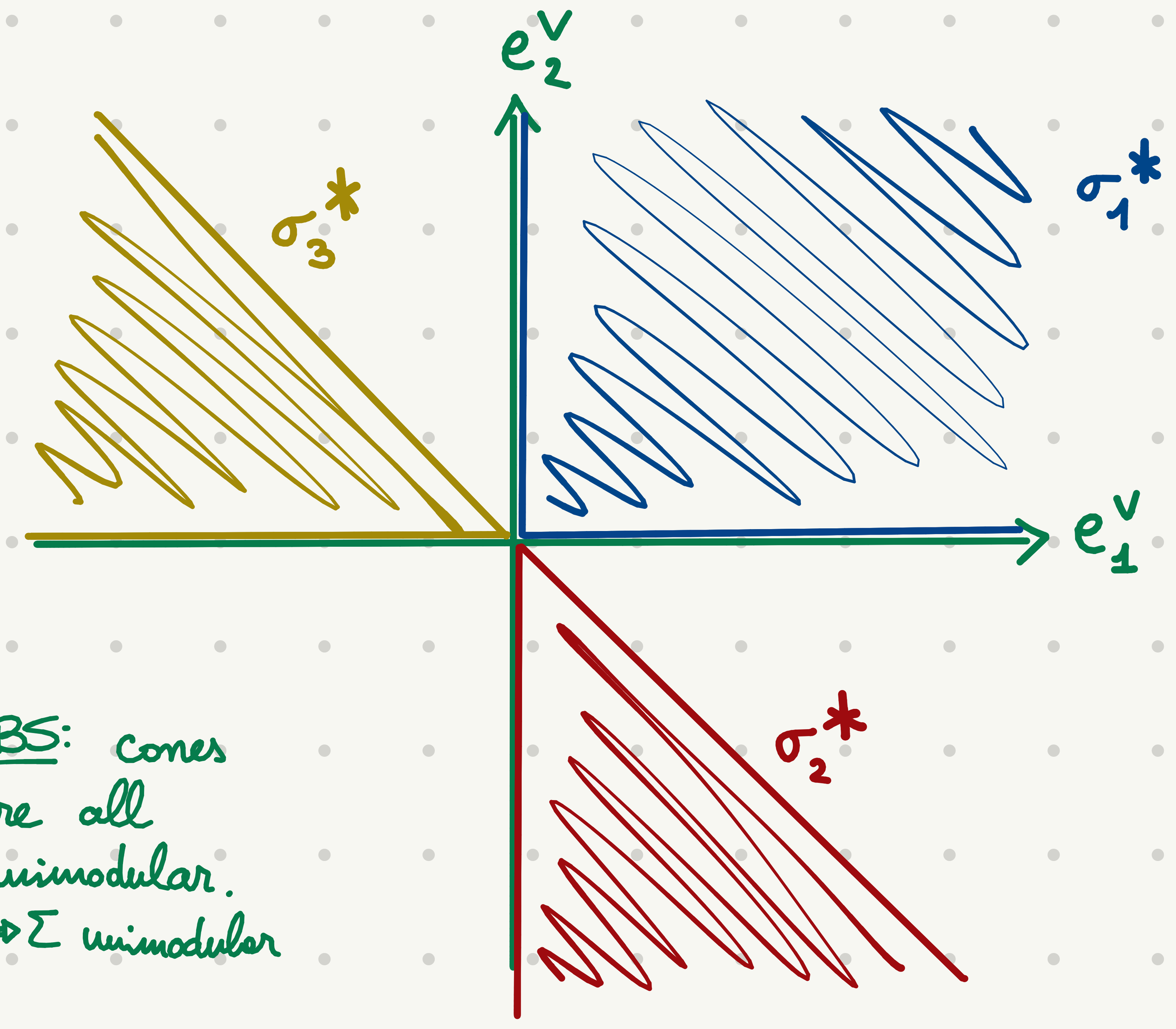
Now, we will make an example...

EXAMPLE $\mathbb{P}^2_{\mathbb{C}}$

Recall that we've constructed $\mathbb{P}^2_{\mathbb{C}}$ the following way:



OBS: cones are all unimodular.
$\Rightarrow \Sigma$ unimodular



Hence, we get the faces:

$$U_{\sigma_1}: \quad \mathbb{C}[\chi^{e_1^*}, \chi^{e_2^*}] =: \mathbb{C}[x_1, y_1]$$
$$U_{\sigma_2}: \quad \mathbb{C}[\chi^{e_1^*-e_2^*}, \chi^{-e_2^*}] =: \mathbb{C}[x_2, y_2]$$
$$U_{\sigma_3}: \quad \mathbb{C}[\chi^{e_2^*-e_1^*}, \chi^{-e_1^*}] =: \mathbb{C}[x_3, y_3]$$

$x_2 = x_1/y_1 \quad y_2 = 1/y_1$

$x_3 = 1/x_2 \quad y_3 = x_2/y_2$

that glue together to make $\mathbb{P}^2_{\mathbb{C}}$. Fixing coordinates $[z_0 : z_1 : z_2]$:

$$\begin{cases} x_1 = \dfrac{z_1}{z_0} \\ y_1 = \dfrac{z_2}{z_0} \end{cases} \qquad \begin{cases} x_2 = \dfrac{z_1}{z_2} \\ y_2 = \dfrac{z_0}{z_2} \end{cases} \qquad \begin{cases} x_3 = \dfrac{z_2}{z_1} \\ y_3 = \dfrac{z_0}{z_1} \end{cases}$$

$$U_0 = \{z_0 \neq 0\} \qquad U_1 = \{z_2 \neq 0\} \qquad U_2 = \{z_1 \neq 0\}$$

Now, we shall compute torus' orbits.

Let $\sigma = e_1$. Therefore $\gamma_{e_1} \in U_{e_1} = U_{\sigma_1} \cap U_{\sigma_2}$.

$$\gamma_{e_1}: S_{e_1} = \mathbb{N}\{\chi^{e_1^\vee}, \chi^{\pm e_2^\vee}\} \longrightarrow \mathbb{C}$$
$$m \longmapsto \begin{cases} 1 & \text{if } m \in e_1^\vee \cap M = \mathbb{Z} e_2^\vee \\ 0 & \text{elsewhere} \end{cases}$$

In coordinates:

$$\gamma_{e_1} = (0,1) \text{ in } \mathbb{C}^2 \simeq U_0$$

hence $\mathcal{O}(e_1) = T \cdot \gamma_{e_1} = \{(0,t) \mid t \in \mathbb{C}\} \subseteq U_0$.

Therefore $F_{e_1} = \overline{\mathcal{O}(e_1)} = \{z_1 = 0\} \subseteq \mathbb{P}^2_{\mathbb{C}}$.

By the analogous argument we get

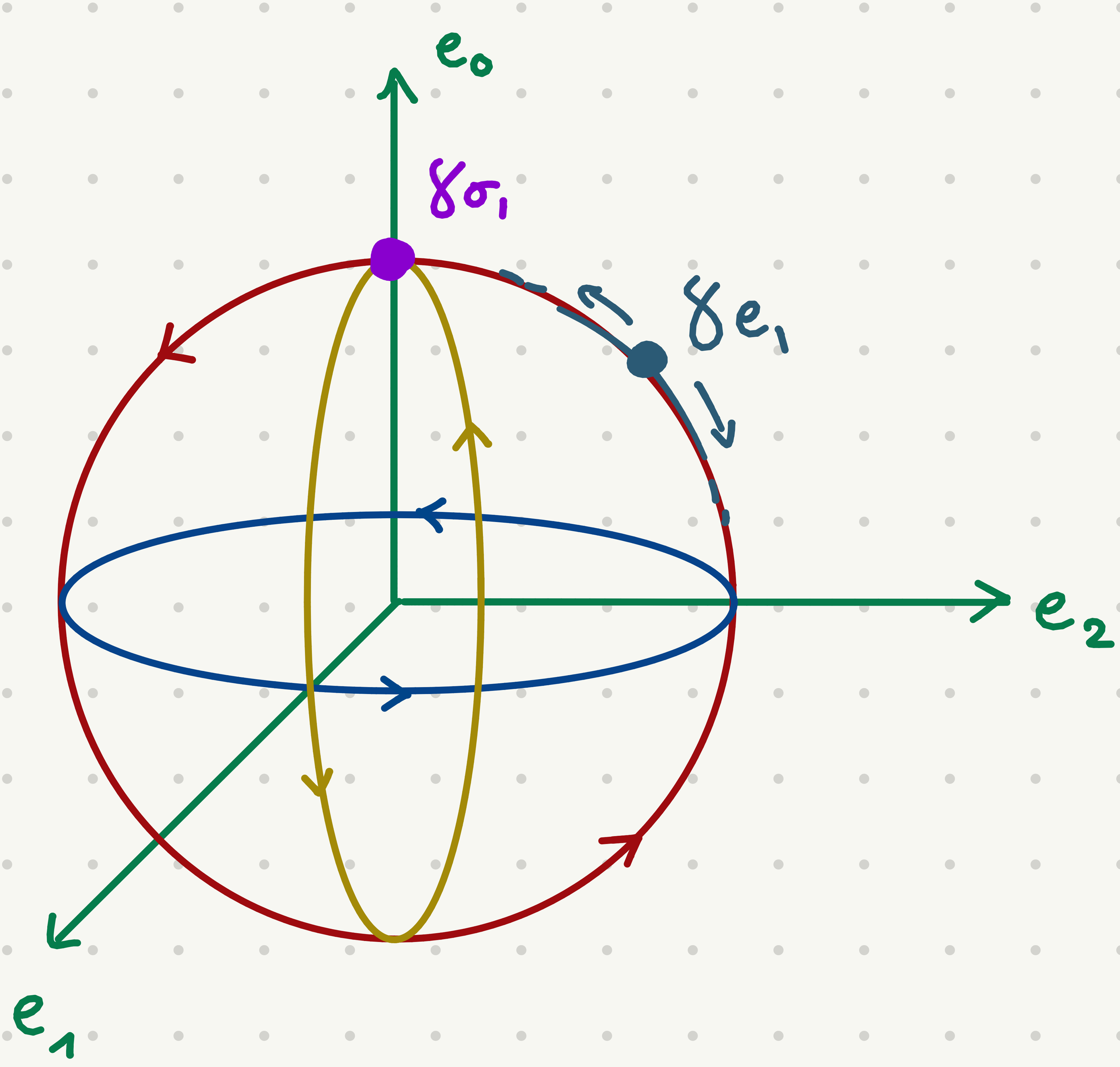
$$F_{e_2} = \overline{\mathcal{O}(e_2)} = \{z_2 = 0\} \quad \text{and} \quad F_{-e_1-e_2} = \{z_0 = 0\}.$$

Considering $\sigma_1 = \mathrm{Cone}\{e_1, e_2\}$, we see that $\sigma_1^\vee = \{0\}$.
Hence in the chart $U_0 \cong U_{\sigma_1}$ we have $\gamma_{\sigma_1} = (0,0) \in U_0$
and so $F_{\sigma_1} = \overline{\mathcal{O}(\sigma_1)} = \{z_1 = z_2 = 0\}$.

Observe that:

$$F_{\sigma_1} = \{z_1 = 0\} \cap \{z_2 = 0\} = F_{e_1} \cap F_{e_2}.$$

Let's try to visualize this. Consider the projection onto the real $\mathbb{P}^2_{\mathbb{R}} \subseteq \mathbb{R}^3$.



Identify the action of the "real torus" as:

$$\begin{aligned} (\mathbb{R}^*)^2 &\longrightarrow \mathbb{R}^3 \\ (\alpha, \beta) &\longrightarrow \big((1,1,1) \to (\alpha, \beta, 1)\big) \end{aligned}$$

Hence, if M is $\mathrm{Span}(e_1^\vee, e_2^\vee)$:

$$\left\{ \begin{aligned} \alpha \cdot e_1 &\longleftrightarrow (\alpha, 0) \\ \alpha \cdot e_2 &\longleftrightarrow (0, \alpha) \\ \alpha \cdot e_3 &\longleftrightarrow (-\alpha, -\alpha) \end{aligned} \right.$$

This happens because $\mathbb{P}^2_{\mathbb{R}} \simeq \mathbb{R}^3 / \mathbb{R}^*$

Now, take $U_0 \simeq U_{\sigma_1}$ in $\mathbb{P}^1_{\mathbb{C}}$. This is the north semisphere.
Given $\gamma_{e_1} := (0,1)$ in U_0, in ▇, the action of the torus lets it move on ◯ $\mapsto F_{e_1} = \{z_1 = 0\}$.
Given $\gamma_{\sigma_1} := (0,0)$ in U_0, in ▇, it cannot move.
$\mapsto F_{\sigma_1} = \{z_1 = z_2 = 0\}$.

▸ THEOREM

Let X_Σ as above. Then the set $\{[F_\sigma]\}_{\sigma\in\Sigma}$ generate $A_*(X)$.

LEMMA Let $Y\subset X$ be a closed subvariety. The following is exact:

$$\begin{array}{ccccccc} A_*(Y) & \longrightarrow & A_*(X) & \longrightarrow & A_*(X\setminus Y) & \longrightarrow & 0 \\ {[V]} & \longmapsto & [V\subseteq X] & \longmapsto & [V\cap(X\setminus Y)] & & \end{array}$$

• Proof We get the following steps:

- $T = F_{\{0\}}$ is an open set in $U_{\{0\}} \subseteq \mathbb{A}^n$. Hence $A_*(T)\cong\langle[T]\rangle$. non-trivial just in A_n
- Let $Y = X\setminus T$. Then by Lemma we have:

$$A_*(X\setminus T) \longrightarrow A_*(X) \longrightarrow A_*(T) \longrightarrow 0$$

Looking at the degrees:

→ degree n: since T dense in X, $\dim(X\setminus T) < \dim(X)$ hence $A_n(X\setminus T) = 0$. By exactness it follows

$$\begin{array}{ccc} A_n(X) & \longrightarrow & A_n(T) \\ {[X]} & \longmapsto & [T] \end{array}$$

is an isomorphism.

→ degree $k<n$: by step 1 we know $A_k(T) = 0$. Therefore $A_k(X\setminus T) \twoheadrightarrow A_k(X)$ surjective. By construction, we know that $X\setminus T \subset \bigcup_{\sigma\neq\{0\}} F_\sigma$. Hence, any cycle in $A_k(X)$ actually lives in the cones $\{F_\sigma\}_{\sigma\neq\{0\}}$.

By an induction on $\dim X$, we conclude. ∎

$\boxed{\text{Chapter 3}}$: $R(X_\Sigma) \cong A^*(X_\Sigma)$

We take the considerations of the previous chapters to describe explicitly $A^*(X_\Sigma)$. Let Σ be smooth (i.e. UNIMODULAR).

Consider for $\Sigma^{(1)} = \{\sigma_1, -, \sigma_k\}$ and identify

$$\sigma_i \longleftrightarrow \text{PRIMITIVE RAY } u_i \in \sigma_i \cap N.$$

Recall also that $F_\sigma \cap U_\tau \neq \emptyset \iff \tau \supseteq \sigma$.

We have the following observations:

1. if $\{i_1, -, i_h\} \in \Sigma$, so they generate a cone, then:

$$F_{u_{i_1}} \cap \_\_ \cap F_{u_{i_h}} = F_{<\sigma_{i_1}, -, \sigma_{i_h}>}.$$

This is true by looking at the orbits $\mathcal{O}(u_i)$ in the affine chart $U_{<\sigma_{i_1}, -, \sigma_{i_h}>}$.

2. if $\{i_1, -, i_h\} \notin \Sigma$, then we get:

$$F_{u_{i_1}} \cap - \cap F_{u_{i_h}} \neq \emptyset \iff F_{u_{i_1}} \cap - \cap F_{u_{i_h}} \cap U_\tau \neq \emptyset \text{ for } \tau \in \Sigma$$
$$\iff \sigma_{i_1}, \dots, \sigma_{i_h} \subseteq \tau \quad \text{↯}$$

hence $F_{u_{i_1}} \cap \_ \cap F_{u_{i_h}} = \emptyset$, given Σ <u>unimodular</u>.

3. consider $m \in M$. Then we have the homomorphism

$$\varphi_m : T \longrightarrow \mathbb{C}$$
$$(t_1, -, t_n) \longmapsto t_1^{m_1} \dots t_n^{m_n}$$

that extends to $\overline{\varphi}_m : X_\Sigma \longrightarrow \mathbb{P}^1_{\mathbb{C}}$. This means:

$$\operatorname{div}(\varphi_m) = \sum_{i=1}^{k} m(u_i)[F_{u_i}].$$

All these relations must be contained in $\operatorname{Rat}(X_\Sigma)$.

Considering the ring structure on $A^*(X_\Sigma)$ we get that:

ⓐ $\rightsquigarrow$ $A^*(X_\Sigma)$ is generated by $[F_{e_i}]$ as a ring.

By associating $[F_{e_i}] \longleftrightarrow X_i$ variable, we get a map: $\mathbb{Z}[X_i] \twoheadrightarrow A^*(X_\Sigma)$.

ⓑ $\rightsquigarrow$ If $\langle e_{i_1}, \ldots, e_{i_k} \rangle \notin \Sigma$ then $F_{e_{i_1}} \cap \ldots \cap F_{e_{i_k}} = \emptyset$ means that $X_{i_1} \ldots X_{i_k}$ goes to 0 in the Chow ring. Such elements identify:

$$\mathbb{Z}\{X_{i_1} \ldots X_{i_k} \mid \{i_1, \ldots, i_k\} \notin \Sigma\} = I(\Delta_\Sigma).$$

ⓒ $\rightsquigarrow$ The divisors of φ^m are in $\mathrm{Rat}(X)$, therefore by the correspondence we can quotient by

$$\mathbb{Z}\{ \sum_i m(e_i) X_i \mid m \in M\} = J.$$

This is the COMBINATORIAL CHOW RING:

$$R(X_\Sigma) = \mathbb{Z}[X_i] / I(\Delta_\Sigma) + J \twoheadrightarrow A^*(X)$$

is a SURJECTIVE RING HOMOMORPHISM.

▸ EXAMPLE $\mathbb{P}^2_{\mathbb{C}}$

Take the previous example. Here we have:

$$F_{e_1} = \{z_1 = 0\} \qquad F_{e_2} = \{z_2 = 0\} \qquad F_{-e_1-e_2} = \{z_0 = 0\}.$$

$$\updownarrow \qquad\qquad \updownarrow \qquad\qquad \updownarrow$$

$$X_1 \qquad\qquad X_2 \qquad\qquad X_3$$

and $R(\Sigma) := \mathbb{Z}[X_1, X_2, X_3] / \langle X_1 X_2 X_3 \rangle + \langle X_1 - X_3, X_2 - X_3 \rangle$.

② We know $F_{e_1} \cap F_{e_2} \cap F_{-e_1-e_2} = \{z_1 = z_2 = z_0 = 0\} = \emptyset$.

③ Consider the relation $X_1 - X_3$ generated when $m = e_1^\vee$.

In this case we can look at the chart U_0 where

$U_0 \equiv U_{\sigma_1} \leftrightarrow \mathbb{C}[x_1, y_1]$ and $x_1 = \frac{z_1}{z_0}$, $y_1 = \frac{z_2}{z_0}$.

Then, since $x_1 \sim \chi^{e_1^\vee}$ and $y_1 \sim \chi^{e_2^\vee}$ we have that

$$\begin{aligned} \varphi_{e_1^\vee} : \mathbb{C}^2 \cong U_{\sigma_1} &\longrightarrow \mathbb{C} \\ (x_1, y_1) &\longmapsto x_1^1 \cdot y_1^0 = x_1 \end{aligned}$$

Extending for $\mathbb{P}^2_{\mathbb{C}}$ with $[z_0 : z_1 : z_2] = [1 : x_1 : x_2]$

$\mathbb{P}^1_{\mathbb{C}}$ with $[u_0 : u_1] = [1 : t]$

we find $\overline{\varphi}_{e_1^\vee} : \mathbb{P}^2_{\mathbb{C}} \longrightarrow \mathbb{P}^1_{\mathbb{C}}$ where $0 \equiv [1:0]$, $\infty \equiv [0:1]$

$$[z_0 : z_1 : z_2] \longmapsto [z_0 : z_1]$$

so $\mathrm{div}(\overline{\varphi}_{e_1^\vee}) = \{z_1 = 0\} - \{z_0 = 0\} = F_{e_1} - F_{-e_1 - e_2}$.

At this point, one can prove the following theorem:

THEOREM (Danilov)

Let Σ be a complete smooth fan. The ring homomorphism:

$$R(\Sigma) \xrightarrow{\sim} A^*(X_\Sigma)$$

is an isomorphism and we also have $A^*(X_\Sigma) \simeq H^*(X_\Sigma, \mathbb{Z})$.

- Proof (idea)

The idea is to show that:

→ $R(\Sigma)$ is torsion-free and it is Cohen-Macaulay.

→ we have an isomorphism $A^*(X_\Sigma) \simeq H^*(X_\Sigma, \mathbb{Z})$ which uses Poincaré Duality (for Σ simplicial, construction works on $\mathbb{Q}$) and the case of Σ projective: SHELLABILITY!

Then one concludes by induction.

→ One observes that $\mathrm{rk}(R(\Sigma))$ and $\mathrm{rk}(H^*(X_\Sigma, \mathbb{Z}))$ are equal, both to $a_n = \#\Sigma^{(n)}$.

⇝ HILBERT POLYNOMIAL for $R(\Sigma)$

⇝ EULER CHARACTERISTIC for $H(X_\Sigma, \mathbb{Z})$

This concludes. □

FULTON's ⊛

Take $\sigma_1, \dots, \sigma_m$ and τ_i = inters. of σ_i with σ_j, $j > i$. Then

(i) $\tau_i \subseteq \sigma_j \iff i \leq j$

(ii) $\Sigma = \coprod [\tau_i, \sigma_i]$

DANILOV

Σ projective fan: g piecewise linear function on Σ.

$x_0 \in N_{\mathbb{R}}$ general position, $\sigma \leq \sigma'$ if $g|_\sigma(x_0) \leq g|_{\sigma'}(x_0)$.