HARMONIC ALGEBRA: a wider picture & the conjecture

▶ A WIDER PICTURE...

After all the work done in the previous episodes, we want to take a step back and see where we stand.

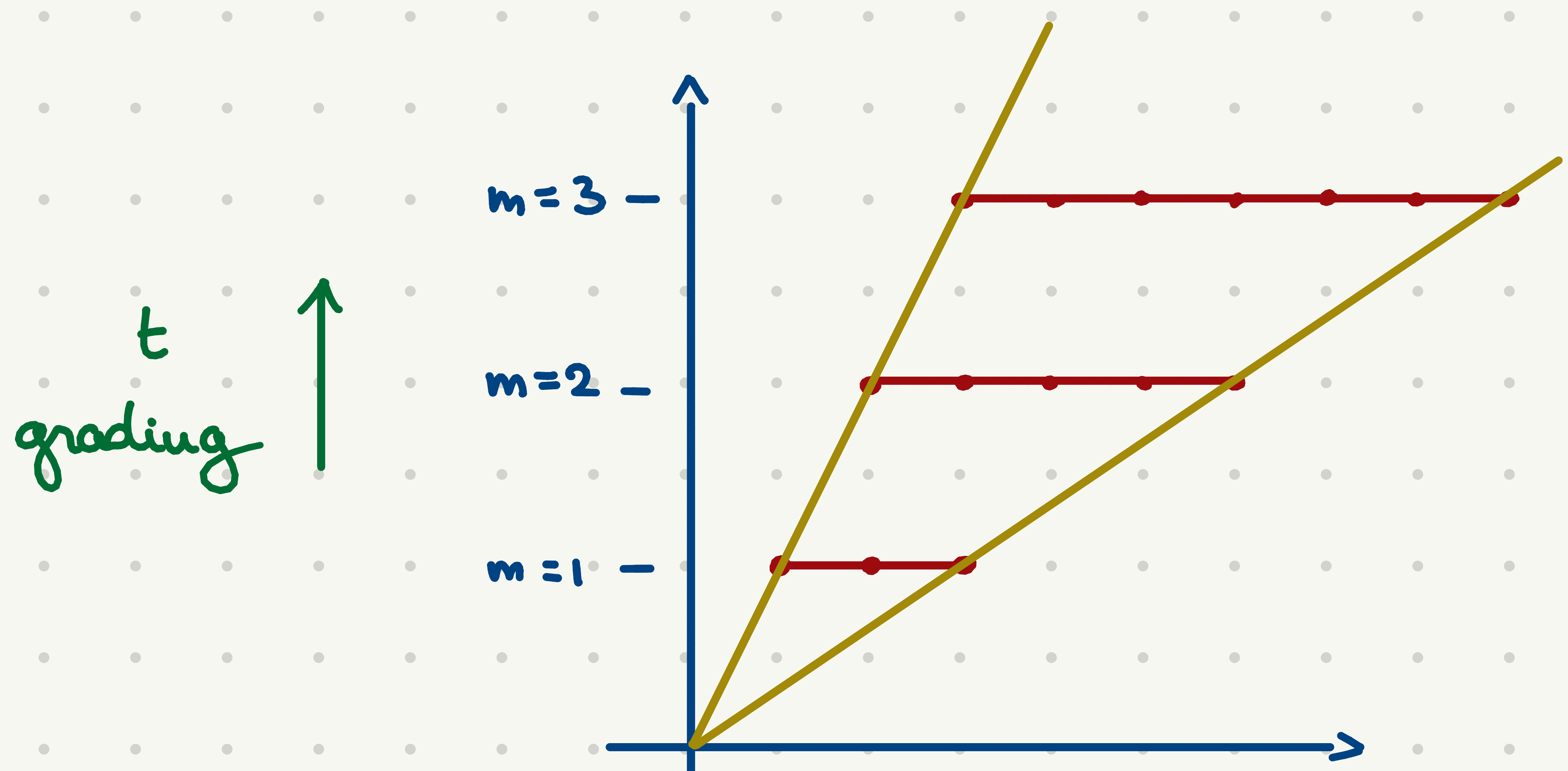
① STARTING POINT: classical Ehrhart theory

Let P be a lattice polytope. We defined the following:

$$\begin{cases} i_P(m) := \# \, mP \cap \mathbb{Z}^n \\ E_P(t) := \sum_{m \geq 0} i_P(m) t^m \end{cases}$$

and Giulia showed us that this series encodes lots of information on P.

The relevant picture here is the following:



which is related to the ring A_P, i.e. $\mathbb{K}[\mathrm{cone}\{P\} \cap \mathbb{Z}^{n+1}]$.

② THE OBJECTIVE: q-analogue

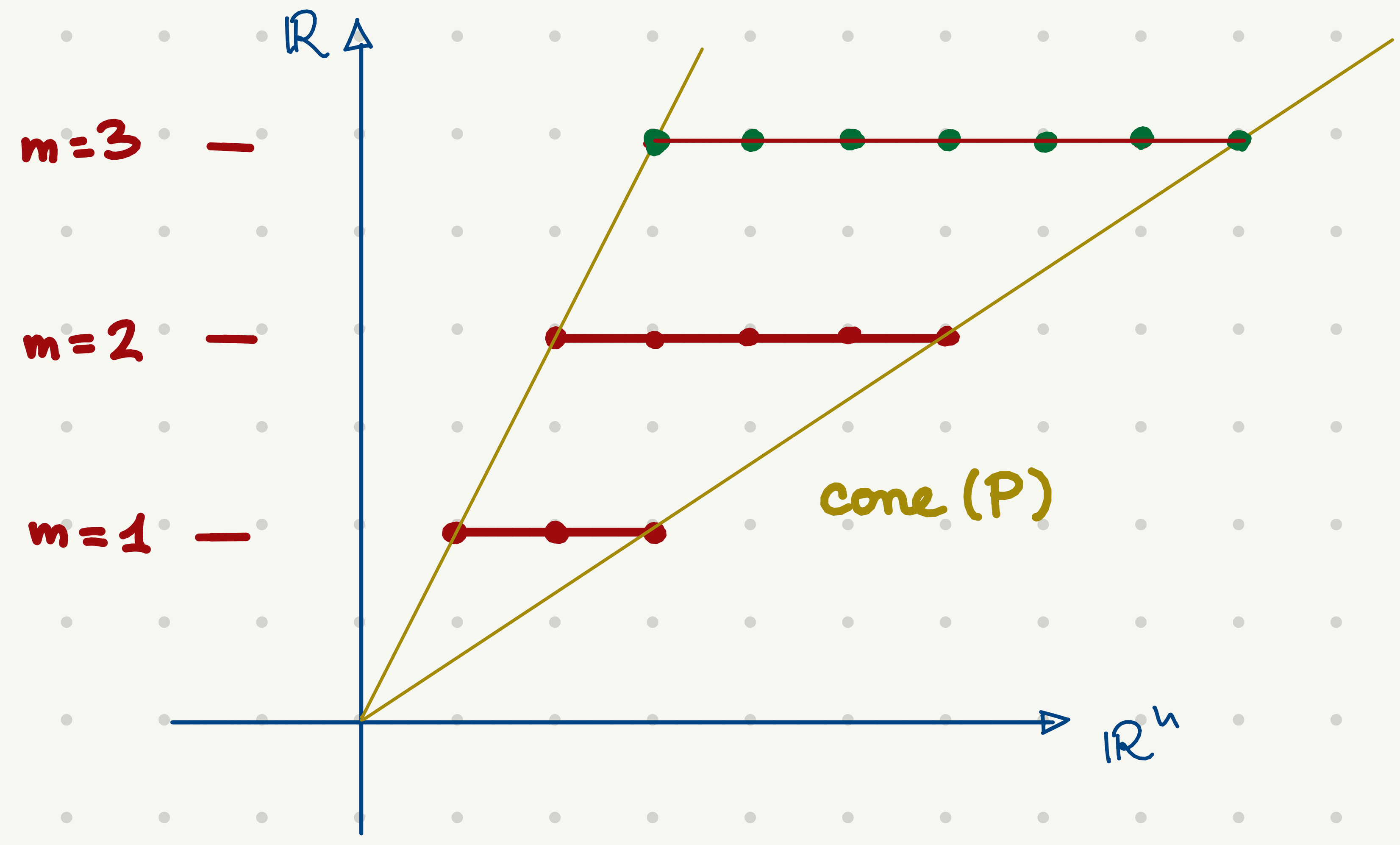
The article by Reiner and Rhoades want to define and study a q-analogue version of this $E_P(t)$.

How do we introduce a q-grading? An algebro-geometric view:

$\mathcal{Z} = mP \cap \mathbb{Z}^n$ as a finite point loci $\dashrightarrow$ $I(\mathcal{Z})$ vanishing ideal $\dashrightarrow$ $\mathrm{gr} I(\mathcal{Z})$ homogeneous version (grading!)

Therefore we can define $R(\mathcal{Z}) := S/_{\mathrm{gr} I(\mathcal{Z})}$ where $S = \mathbb{K}[x_1, \dots, x_n]$.

▸ EXAMPLE Line Segment



We start by the same picture of the classical version.

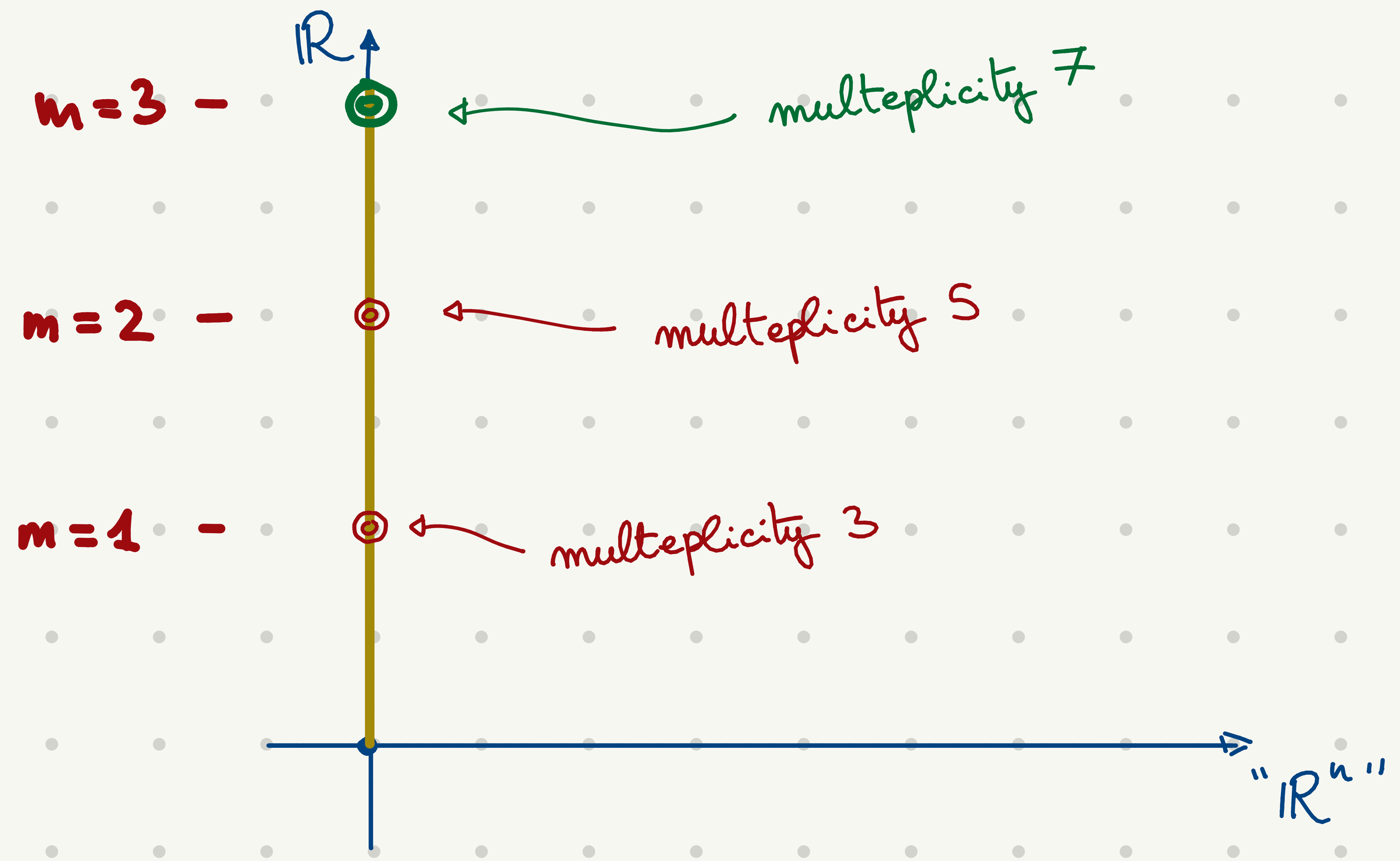
Take for example $m = 1$:

$$\mathcal{Z} = P \cap \mathbb{Z}^n \times \{1\}$$

then $\mathcal{Z} = \{1, 2, 3\} \times \{1\}$.

$$I(\mathcal{Z}) = \big((x-1)(x-2)(x-3) \big).$$

$\Downarrow$ Going to $\mathrm{gr}\, I$...



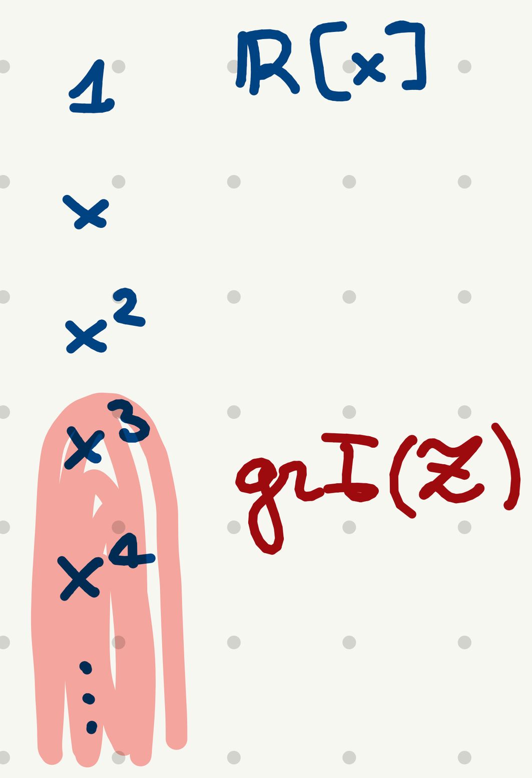
Taking the graded part:

$$\mathrm{gr}\, I(\mathcal{Z}) = (x^3)$$

so the point locus gets deformed to the origin, with multeplicity.

By taking the quotient $R(\mathcal{Z}) = \mathbb{R}[x] / (x^3)$ we can look at the degrees and we get

$$i_P(1; q) = 1 + q + q^2.$$



Analogously $i_P(m; q) = 1 + q + \ldots + q^{2m+1}$.

The naïve approach is to define an analogue version of A_P, so $R(\mathrm{cone}\{P\} \cap \mathbb{Z}^{n+1})$ but this does not work:

→ $\mathrm{cone}\{P\} \cap \mathbb{Z}^{n+1}$ is Zariski dense i.e. $I(\mathrm{cone}\{P\} \cap \mathbb{Z}^{n+1}) = (0)$.

→ we lose the "t" grading that we got from taking the cone...

This is why we work with $\bigoplus_{m \geq 0} R(mP \cap \mathbb{Z}^n)$:

$$\begin{cases} i_P(m; q) := \mathrm{Hilb}\big(R(mP \cap \mathbb{Z}^n), q\big) \\ E_P(t, q) = \sum_{m \geq 0} i_P(m, q)\, t^m. \end{cases}$$

3 MULTIPLICATION & MINKOWSKI SUM

Thus, we have a new necessity: in A_P we had a semigroup operation on cone{P} translating onto multiplication in A_P.

$(\text{cone}\{P\} \cap \mathbb{Z}^{n+1}, +)$ is a semigroup so $A_P = \mathbb{R}[\text{cone}\{P\} \cap \mathbb{Z}^{n+1}]$

$$\underline{x}^{\underline{z}+\underline{z}'} = \underline{x}^{\underline{z}} \cdot \underline{x}^{\underline{z}'}.$$

Here:

"Minkowski sum" on $\mathcal{Z}+\mathcal{Z}'$ $\xrightarrow[\text{translation}]{\text{Algebraic}}$ Product onto Harmonic Spaces!

We can breafly recall:

$$P \to mP \cap \mathbb{Z}^n \to \operatorname{gr} I(mP \cap \mathbb{Z}^n) \to \begin{cases} R(mP \cap \mathbb{Z}^n) = S/_{\operatorname{gr} I(mP \cap \mathbb{Z}^n)} \\ \quad \| \text{ same numerology!} \\ V_{mP \cap \mathbb{Z}^n} = \operatorname{gr} I(mP \cap \mathbb{Z}^n)^{\perp} \end{cases}$$

: the numerology stays the same: we have $\mathbb{K}$-isomorphisms between

$R(\mathcal{Z})$ "$\cong$" $V(\mathcal{Z})$ i.e. $S/_{\operatorname{gr} I(\mathcal{Z})}$ "$\cong$" $\operatorname{gr} I(\mathcal{Z})^{\perp}$.

Hence $\operatorname{Hilb}(R(\mathcal{Z}); q) = \operatorname{Hilb}(V_{\mathcal{Z}}; q)$.

: in Harmonic spaces $\mathbb{D}$ and $\hat{\mathbb{D}}$ we have a nice "exponential" basis for $I(\mathcal{Z})^{\perp} \subseteq \mathbb{D}^{\perp}$, so $I(\mathcal{Z})^{\perp} \cdot I(\mathcal{Z}')^{\perp} = I(\mathcal{Z}+\mathcal{Z}')^{\perp}$. (For $\mathcal{Z}, \mathcal{Z}'$ finite point loci!)

Given these two observations, finally ...

4 HARMONIC ALGEBRA

We can rewrite $\bigoplus_{m\geq 0} R(mP\cap\mathbb{Z}^n)$ with the following more suitable structure ...

DEF. Harmonic Algebra $\mathcal{H}_P$

Let P be a lattice polytope in $\mathbb{R}^n$. Let

$$\mathbb{R}[y_0, \underline{y}] := \mathbb{R}[y_0, y_1, \dots, y_n] \cong \mathbb{R}[y_0] \otimes \mathbb{D}_{\mathbb{R}}[\underline{y}].$$

as a polynomial ring with the following $\mathbb{N}^2$-bigraded structure :

$$\deg(y_0) = (1,0) \qquad \text{and} \qquad \deg(y_i) = (0,1) \quad \forall i=1,\dots,n.$$

Then the HARMONIC ALGEBRA associated to P is a $\mathbb{R}$-subspace

$$\mathcal{H}_P := \bigoplus_{m=0}^{\infty} \mathbb{R}y_0^m \otimes_{\mathbb{R}} V_{mP\cap\mathbb{Z}^n}$$

and its INTERIOR IDEAL is

$$\overline{\mathcal{H}}_P := \bigoplus_{m=0}^{\infty} \mathbb{R}y_0^m \otimes_{\mathbb{R}} V_{\mathrm{int}(mP)\cap\mathbb{Z}^n}.$$

PROPOSITION Algebra $\mathcal{H}_P$

The $\mathbb{R}$-linear subspace $\mathcal{H}_P \subseteq \mathbb{R}[y_0, \underline{y}]$ has an $\mathbb{N}^2$-bigraded algebra structure and $\overline{\mathcal{H}}_P$ is an ideal of $\mathcal{H}_P$.

Furthermore, we get that by setting $y_0^{m_0} y_1^{m_1} \dots y_n^{m_n} \mapsto t^{m_0} q^{m_1+\dots+m_n}$:

$$\begin{cases} \mathrm{Hilb}(\mathcal{H}_P; t,q) = E_P(t,q) \\ \mathrm{Hilb}(\overline{\mathcal{H}}_P; t,q) = \overline{E}_P(t,q). \end{cases}$$

LEMMA

Given finite subsets $\mathcal{Z} \subseteq \mathcal{Z}' \subseteq \mathbb{K}^n$ one has $V_{\mathcal{Z}} \subseteq V_{\mathcal{Z}'}$.

- **DIM. Lemma**

Simply $\mathcal{Z} \subseteq \mathcal{Z}' \Rightarrow I(\mathcal{Z}) \supseteq I(\mathcal{Z}') \Rightarrow \mathrm{gr}\, I(\mathcal{Z}) \supseteq \mathrm{gr}\, I(\mathcal{Z}') \Rightarrow V_{\mathcal{Z}} \subseteq V_{\mathcal{Z}'}$. ∎

- DIM. Proposition
  - $\mathcal{H}_P$ is an algebra: we need to check that
    $$(\mathbb{R}y_0^m \otimes V_{mP\cap\mathbb{Z}^n}) \cdot (\mathbb{R}y_0^{m'} \otimes V_{m'P\cap\mathbb{Z}^n}) \subseteq \mathbb{R}y_0^{m+m'} \otimes V_{(m+m')P\cap\mathbb{Z}^n}$$
    so specifically it suffices to see
    $$V_{mP\cap\mathbb{Z}^n} \cdot V_{m'P\cap\mathbb{Z}^n} \subseteq V_{mP\cap\mathbb{Z}^n + m'P\cap\mathbb{Z}^n} \quad \text{(nice behaviour w.r. to Minkowski sum)}$$
    and that by the previous Lemma
    $$V_{mP\cap\mathbb{Z}^n + m'P\cap\mathbb{Z}^n} \subseteq V_{(m+m')P\cap\mathbb{Z}^n} \quad \text{which concludes.}$$
  - $\overline{\mathcal{H}}_P$ is an ideal: analogous.
  - Hilbert functions: they follow from the equalities on $\mathbb{N}\times\{0\}$ grading
    $$\mathrm{Hilb}(V_{mP\cap\mathbb{Z}^n}; q) = \mathrm{Hilb}(\mathrm{gr}\,I(mP\cap\mathbb{Z}^n)^\perp; q) = \mathrm{Hilb}(R(mP\cap\mathbb{Z}^n); q). \quad \blacksquare$$

... AND THE CONJECTURE

Finally, we have a well-defined algebraic structure on which to try proving analogous results to the Classical Ehrhart Theory.
Recall how Stanley proved algebraically properties for $E_P(t)$:

> RATIONALITY of $E_P(t)$:
>
> A_P is finitely generated $\mathbb{K}$-algebra $\Rightarrow$ $\mathrm{Hilb}(A_P, t)$ is rational

> DENOMINATOR $(1-t)^{d+1}$:
>
> via Noether's Normalization Lemma (graded):
>
> $\exists\, \theta_i$ s.o.p. $\mid$ A_P is a $\mathbb{K}[\theta_1, \dots, \theta_{d+1}]$ - finitely generated module.

> NON-NEGATIVITY of $\{h_i^*\}$:
>
> by showing A_P is Cohen-Macaulay (Artinian reduction for A_P!).

It would be nice to transfer these nice properties on $\mathcal{H}_P$.

- **CONJECTURE (1.1)** ~ The q-Ehrhart "Theorem"

Let P be a d-dimentional lattice polytope in $\mathbb{R}^n$. Then:

(a) $E_P(t,q) = \dfrac{N(t,q)}{D(t,q)}$ and $\bar{E}_P(t,q) = \dfrac{\bar{N}(t,q)}{D(t,q)}$ where the denominator is $D(t,q) = \prod_{i=1}^{\nu} (1 - t^{a_i} q^{b_i})$ with $\nu \geq d+1$.

(b) the numerators $N(t,q), \bar{N}(t,q) \in \mathbb{Z}[t,q]$.

(c) if P is a simplex and $\nu = d+1$ then the coefficients of $N(t,q)$ and $\bar{N}(t,q)$ are non-negative.

In suport of this, we can expect...

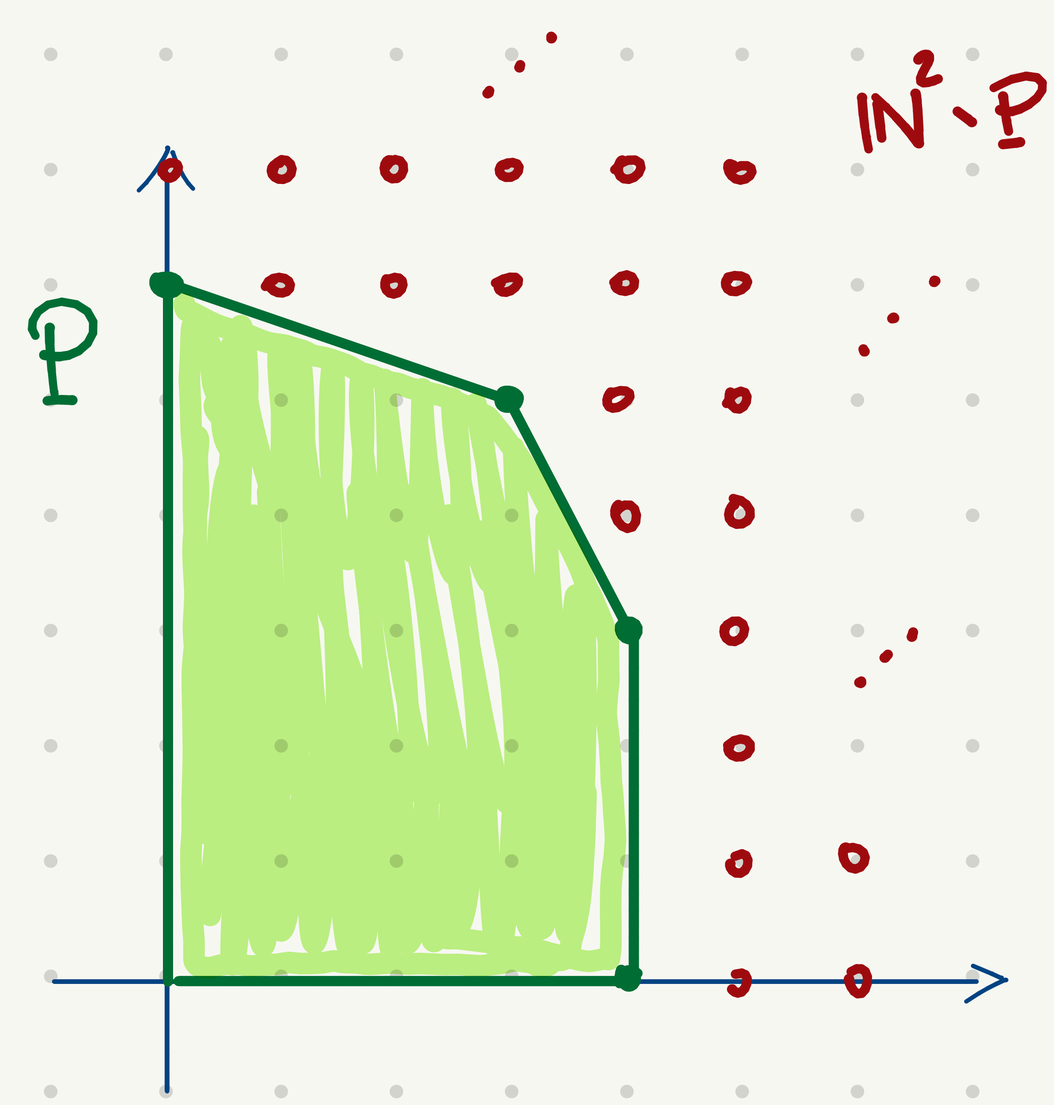
- **CONJECTURE (5.5)** ~ The "nice properties"

Let P be any lattice polytope. The harmonic algebra is:

(a) a Noetherian finitely-generated $\mathbb{R}$-subalgebra of $\mathbb{R}[y_0, y]$.

(b) a Cohen-Macaulay algebra.

5 THE ANTIBLOCKING-POLYTOPE CASE



- **DEF.** Antiblocking Polytope

A polytope $P \subset \mathbb{R}^n_{\geq 0}$ is ANTIBLOCKING if

$\underline{0} \leq \underline{z}' \leq \underline{z}$ and $\underline{z} \in P \implies \underline{z}' \in P$.

- **DEF.** Shifted Point Locus

A subset $\mathcal{Z} \subset \mathbb{N}^n$ is SHIFTED if it forms a lower order ideal on $\mathbb{N}^n$, i.e.:

$\mathrm{span}_{\mathbb{R}}(\underline{x}^{\underline{a}} \mid \underline{a} \in \mathbb{N}^n \setminus \mathcal{Z})$ is a monomial ideal of $S = \mathbb{K}[\underline{x}]$.

In this case, the ideal $\mathrm{gr} I(\mathcal{Z})$ and $V_{\mathcal{Z}}$ are nicely described:

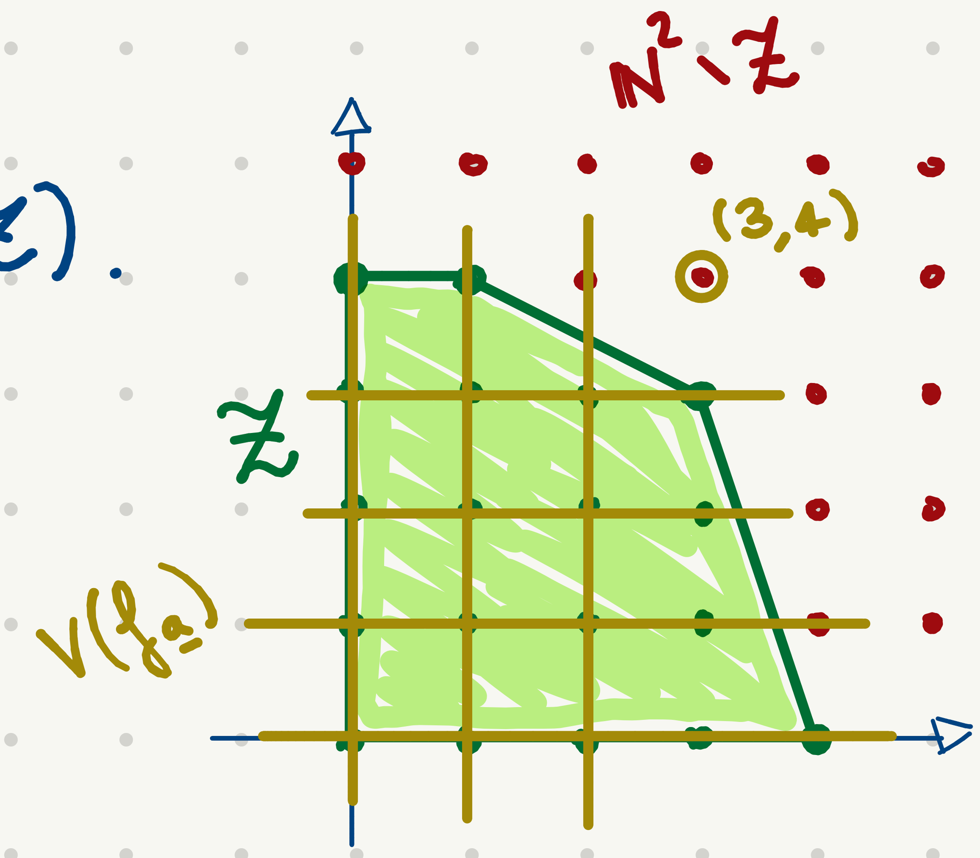
LEMMA

Let $\mathcal{Z} \subseteq \mathbb{N}^n$ be a finite point locus. Then:

(i) the ideal $\operatorname{gr} I(\mathcal{Z}) \subset S$ is monomial with basis $\{ \underline{x}^{\underline{a}} \mid \underline{a} \notin \mathcal{Z} \}$.

(ii) the harmonic space $V_{\mathcal{Z}} \subseteq S$ has an $\mathbb{R}$-basis $\{ \underline{y}^{\underline{a}} \mid \underline{a} \in \mathcal{Z} \} \subseteq \mathbb{D}$.

• DIM.



(i) We show that $I := \operatorname{span}_{\mathbb{K}} \{ x^{\underline{a}} \mid \underline{a} \notin \mathcal{Z} \} \subseteq \operatorname{gr} I(\mathcal{Z})$.

Fixed $\underline{a} \notin \mathcal{Z}$ let

$$f_{\underline{a}}(\underline{x}) = \prod_{i=1}^{n} x_i (x_i - 1) \dots (x_i - a_i + 1).$$

Then for all $\underline{z} \in \mathcal{Z}$ we have $\underline{a} \not\leq \underline{z}$ which means

$\exists\, i$ such that $z_i < a_i \implies f_{\underline{a}}(\underline{z}) = 0$.

Therefore $f_{\underline{a}}(\underline{x}) \in I(\mathcal{Z})$ so $\tau(f_{\underline{a}}) = \underline{x}^{\underline{a}} \in \operatorname{gr} I(\mathcal{Z})$.

Then since $S/_{I} \twoheadrightarrow S/_{\operatorname{gr} I(\mathcal{Z})}$ we conclude by $\mathbb{R}$-dimension.

(ii) We just need to show that $\underline{y}^{\underline{a}} \in V_{\mathcal{Z}}$ $\forall\, \underline{a} \in \mathcal{Z}$:

$$\forall\, \underline{b} \notin \mathcal{Z} \quad \underline{b} \not\leq \underline{a} \implies \underline{x}^{\underline{b}} \odot \underline{y}^{\underline{a}} = 0.$$

By dimension we conclude. ∎

Since the descriptions of A_P and $\mathcal{H}_P$ are nice, we get that...

PROPOSITION

Let $P \subset \mathbb{R}^n$ be an antiblocking lattice polytope. One has an $\mathbb{N}^2$-graded algebra isomorphism $A_P \xrightarrow{\sim} \mathcal{H}_P$ induced by the

$$y_0^m \underline{y}^{\underline{z}} \longmapsto y_0^m \otimes \underline{y}^{\underline{z}}$$

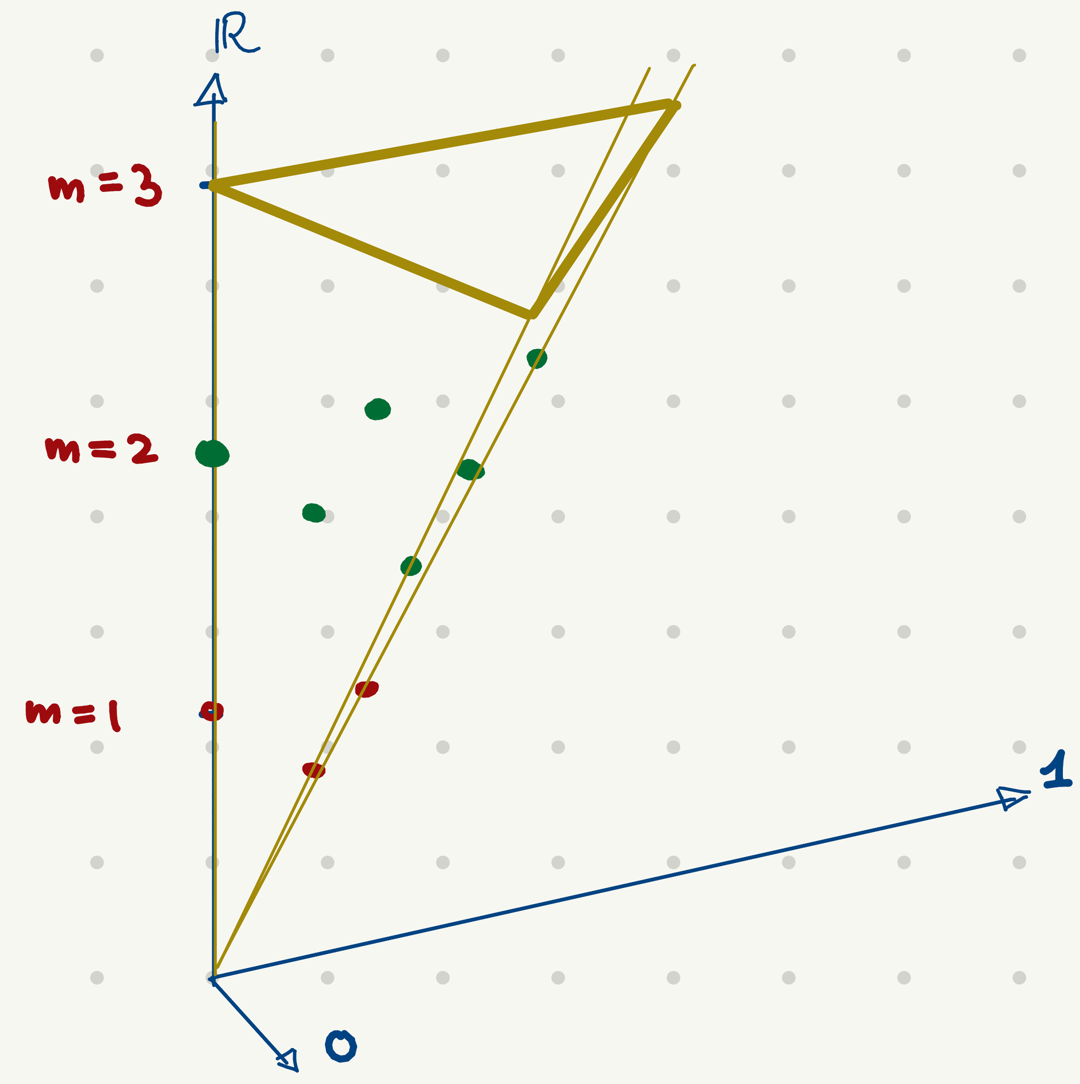
identification $\mathbb{R}[y_0, \underline{y}] \cong \mathbb{R}[y_0] \otimes \mathbb{D}_{\mathbb{R}}(\underline{y})$.

$\implies$ Conjecture 5.5 holds for P antiblocking lattice polytope.

② EXAMPLE An antiblocking polytope $n = d = 2$

Let's consider a 2-dimensional polytope, with the property of being anti-blocking. In this case, take

$$\mathcal{Z} = 1 \cdot P \cap \mathbb{Z}^2$$



then:

- $$I(\mathcal{Z}) = \begin{pmatrix} x_0x_0(x_0-1),\ x_0x_0x_1,\ x_0(x_1-1)(x_0-1),\ x_0(x_1-1)x_1, \\ x_1x_0(x_0-1),\ x_1x_0x_1,\ x_1(x_1-1)(x_0-1),\ x_1(x_1-1)x_1 \end{pmatrix} \overset{!}{=} \big(x_0x_1,\ x_0(x_0-1),\ x_1(x_1-1)\big)$$
- $$\mathrm{gr}I(\mathcal{Z}) = \left(x_0^2,\ x_0x_1,\ x_1^2\right)$$

$$\boxed{\begin{array}{ccc} & 1 & \\ x_0 & & x_1 \\ \cancel{x_0^2} & \cancel{x_0x_1} & \cancel{x_1^2} \end{array}}$$

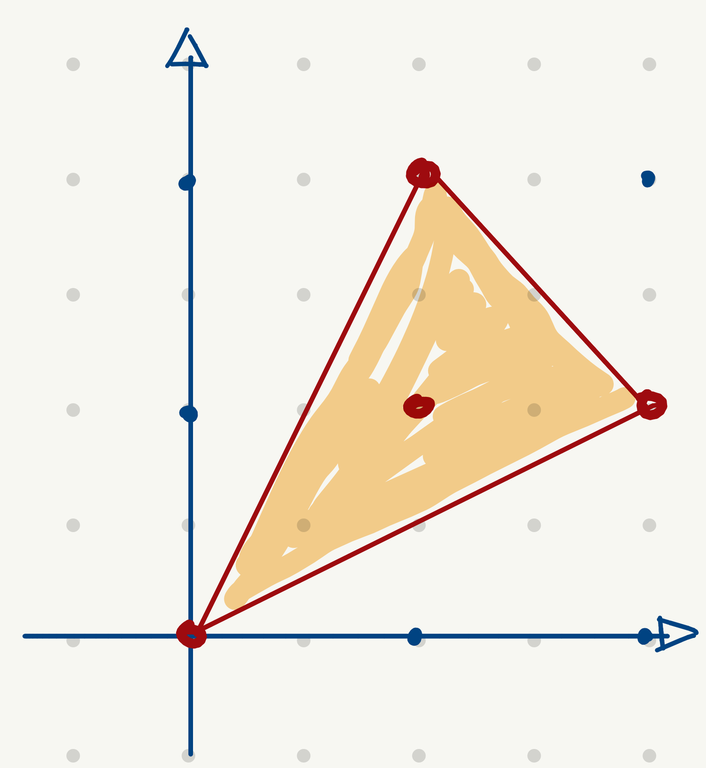
- $$R(\mathcal{Z}) = \mathbb{K}[x_0, x_1] \big/ \mathrm{gr}I(\mathcal{Z}) = \mathrm{span}_{\mathbb{K}}\{1, x_0, x_1\}$$
- $$\mathrm{gr}I(\mathcal{Z})^{\perp} = \{ g \in \mathbb{D} \mid \langle f, g\rangle = 0 \ \ \forall f \in \mathrm{gr}I(\mathcal{Z}) \} \overset{!}{=} \mathrm{span}_{\mathbb{K}}\{1, x_0, x_1\}$$

because everything in 2^{nd} degree has $f \in \mathrm{gr}I(\mathcal{Z})$ s.t. $\langle f, g\rangle \neq 0$.

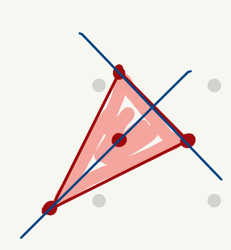
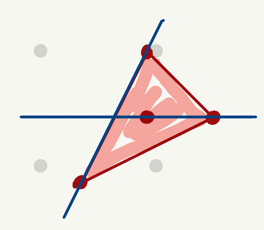
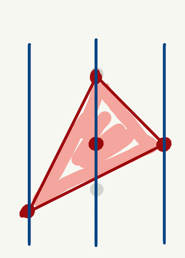
These anti-blocking are nice because in general...

③ EXAMPLE $\mathrm{gr}I$ is not monomial!

Consider the following polytope in $\mathbb{R}^2$: there is no affine transformation of $\mathbb{Z}^2$ that sends P to an antiblocking polytope.



$$\mathrm{gr}I(P \cap \mathbb{Z}^2) = \left(x_1^2 - x_2^2,\ 2x_1x_2 - x_2^2,\ x_2^3\right)$$

$\overset{!}{=}$ ( ,  , )

$$V_{\mathcal{Z}} = \mathrm{gr}I(P \cap \mathbb{Z})^{\perp} = \mathrm{span}_{\mathbb{K}}\{1,\ y_1,\ y_2,\ y_1^2 + y_1y_2 + y_2^2\}$$