TEOREMA di SCHOTTY
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PICCOLO TEOREMA di PICARD

$f : \Delta_r \to \mathbb{C}$ olomorfa $\quad \Delta_r = r\Delta$

def: Il numero di LANDAU per $f : \Delta_R \to \mathbb{C}$ olomorfa

$$L_f = \sup \{ r > 0 \mid f(\Delta_R) \supset \Delta_r \}$$

LEMMA: Dato $R > 0$, $f : \Delta_R \to \mathbb{C}$ olomorfa $f(0) = 0$

$$f(\Delta_R) \supseteq \Delta_{\frac{R \cdot |f'(0)|^2}{4 \|f\|_\infty}}$$

$$\mathcal{R} = \{ f : \Delta \to \mathbb{C} \text{ olom.} \mid f(0) = 0 \,,\, f'(0) = 1 \}$$

def: Il numero di LANDAU

$$L = \inf \{ L_f : f \in \mathcal{R} \}$$

TEO: (BLOCH - LANDAU) $L \geq 1/16 \,,\; f \in \mathcal{O}(\Delta) \,,\, f'(0) = 1$

LEMMA: $f : \Omega \to \Omega_{0,1}$ olomorfa $\Omega \subseteq \mathbb{C}$ sempl connesso

$\Omega_{0,1} := \mathbb{C} \setminus \{0,1\} \Rightarrow \exists\, g \in \mathcal{O}(\Omega)$

$$f = -\exp(\pi i \cosh(2g))$$

1) $g(\Omega)$ non contiene nessun disco di raggio 1

2) $K \subset\subset \Omega_{0,1}$, $\exists\, C_0 := C_0(K)$: $f(0) \in K$

$$|g(0)| < C_0$$

OSS: $f : \Omega \to \Omega_{0,1}$, $|\log|f|| = \pi \sinh(2|\mathrm{Re}\, g|)|\sin(2\,\mathrm{Im}\, g)|$

TEO: (SCHOTTKY) Dato $0 < r < 1$, $a_0 \in \Omega_{0,1} \Rightarrow C_0 := C_0(r, a_0)$

$\forall f : \Delta \to \Omega_{0,1}$ olomorfa, $\forall z \in \Delta_r$, $f(0) = a_0$

$$|f(z)| \leq C_0$$

C_0 è limitata sui cpt di $[0,1) \times \Omega_{0,1}$

DIM: $g \in \mathcal{O}(\Delta)$ $\quad f = -\exp(\pi i \cosh(2g))$. $r < R \leq 1$

$z_0 \in \Delta_r$: $|g'(z_0)| \neq 0$

$$\varphi(z) = \frac{g(z_0 + (1 - r/R) z)}{(1 - r/R)\, g'(z_0)}$$

1) φ è olomorfa in Δ_R $\quad z \in \Delta_R$

$$|z_0 + (1 - r/R) z| \leq |z_0| + 1 - \frac{r}{R} |z|$$

$$\leq r + \frac{R-r}{\cancel{R}} \cancel{R} \leq R \leq 1$$

2) $$\varphi'(z) = \frac{g'(z_0 + (1 - r/R) z) \cdot (1 - r/R)}{(1 - r/R)\, g'(z_0)}$$

$$\varphi'(0) = 1$$

$$16 \leq \frac{1}{(1 - r/R) |g'(z_0)|}$$

$$|g'(z_0)| \leq \frac{16}{1 - r/R}$$

$\forall z \in \Delta_r$

$$|g(z) - g(0)| = \left| \int_0^z g'(\xi)\, d\xi \right| \leq \frac{16 r}{1 - r/R}$$

$$|g(z)| \leq |g(0)| + \frac{16 r}{1 - r/R} \qquad R \to 1$$

$\forall z \in \Delta_r \qquad |f(z)| \leq \exp\left(\pi \sinh\left(2|g(0)| + \frac{32r}{1-r}\right)\right) := C_0(r, a_0)$

$$K \subset\subset [0,1) \times \Omega_{0,1} \qquad p_1, p_2$$

$$p_2(K) = K_r \subset\subset \Omega_{0,1}, \quad C_0(K_r)$$

$$C_0(K_r) : p_1(K) \to \mathbb{R}_+$$

TEO: (LANDAU) Dati $(a_0, a_1) \in \mathbb{C} \times \mathbb{C}^* \Rightarrow \exists R(a_0, a_1) > 0 \ \nexists r :$

$\exists f \in Hol(\Delta_r, \Omega_{0,1})$

$$r < R(a_0, a_1)$$

TEO: (PICCOLO TEOREMA di PICARD) $f : \mathbb{C} \to \Omega_{a,b}$ olomorfa

$\Rightarrow$ f è costante $\quad b \neq a$

DIM: $\dfrac{f-a}{b-a}$, $\quad f : \mathbb{C} \to \Omega_{0,1}$, $\exists z_0 \in \mathbb{C}$

$$a_0 = f(z_0) \neq 0, 1, \quad \underset{\displaystyle a_1}{\underset{\|}{f'(z_0)}} \neq 0$$

$r \in [|z_0|, +\infty) \qquad r \leq R(a_0, a_1).$

COR: $f: \mathbb{C} \to \mathbb{C}$ olomorfa, $f \circ f$ ha un pt fisso oppure f è una traslazione

DIM: $f \circ f$, f

$$g(z) = \frac{f(f(z)) - z}{f(z) - z}$$

$g: \mathbb{C} \to \{0,1\}$, $g(z) = c \neq 0, 1$

$$f(f(z)) - z = c\,(f(z) - z)$$

$$f'(f(z)) \cdot f'(z) - 1 = c\,(\underline{f'(z)} - 1)$$

$$f'(z)\,(f'(f(z)) - c) = 1 - c$$

f' non ha zeri $f' \neq c$ $\qquad$ f' è costante

$f = az + b \qquad a = 1 \qquad b \neq 0$

$f' \circ f$

$$\mathbb{C}^* \xrightarrow{f} \Omega_{0,1}$$

$$\begin{tikzcd}
\mathbb{C} \arrow[d, "p"'] \arrow[dr, "g"] & \\
\mathbb{C}^* \arrow[r, "f"'] & \Omega_{0,1}
\end{tikzcd}$$

$g = f \circ p$

$$\Delta \xrightarrow{p} \Omega_{0,1}$$

$$\begin{tikzcd}
 & \Delta \arrow[d, "p"] \\
\mathbb{C} \arrow[ur, "g"] \arrow[r, "f"'] & \Omega_{0,1}
\end{tikzcd}$$

D

$\underbrace{p \circ g} = f$

$\mathbb{C} \setminus \{0,1\}$

p